ECE531 Homework Assignment Number 6

Solution

Make sure your reasoning and work are clear to receive full credit for each problem. Note that, throughout this solution, the notation $\pi_y(\theta)$ is equivalent to $w(\theta|y)$ and the notation $\pi(\theta)$ is equivalent to $w(\theta)$.

1. 5 points. Poor textbook Chapter IV, Problem 1. Also try to find the MMAE estimator (this is called the ABS estimator in the Poor textbook).

   Solution: To solve these problems, you just need to compute the posterior density

$$\pi_y(\theta) \quad = \quad \frac{p_\theta(y)\pi(\theta)}{p(y)} = \frac{e^{-\theta|y|}}{\int_1^e e^{-\theta|y|}\,d\theta}$$

   where we have canceled the common term $\frac{1}{2}$ and where $\pi(\theta) = w(\theta)$ in the Poor textbook notation.

   (a)

$$\hat{\theta}_{MAP}(y) = \arg\max_{1\le\theta\le e}\left(\pi_y(\theta)\right) = \arg\max_{1\le\theta\le e}\left(-\theta|y|\right) = 1.$$

   (b)

$$\hat{\theta}_{MMSE}(y) = \frac{\int_1^e \theta e^{-\theta|y|}d\theta}{\int_1^e e^{-\theta|y|}d\theta} = \frac{1}{|y|} + \frac{e^{-|y|} - e^{1-e|y|}}{e^{-|y|} - e^{-e|y|}}.$$

   (c) We need to find $\hat{\theta}_{MMAE}$ such that

$$\frac{\int_1^{\hat{\theta}_{MMAE}} e^{-\theta|y|}d\theta}{\int_1^e e^{-\theta|y|}d\theta} = \frac{1}{2}.$$

   A bit of calculus and algebra yields the desired result

$$\hat{\theta}_{MMAE} = \frac{\ln(e^{-|y|e} + e^{-|y|})}{-|y|} + \frac{\ln 2}{|y|}.$$

2. 5 points. Poor textbook Chapter IV, Problem 3.

   Solution: The Bayes approach requires us to compute the posterior density $\pi_y(\theta)$.

$$\pi_y(\theta) = w(\theta|y) = \frac{\theta^y e^{-\theta}e^{-\alpha\theta}}{\int_0^\infty \theta^y e^{-\theta}e^{-\alpha\theta}d\theta} = \frac{\theta^y e^{-(\alpha+1)\theta}(1+\alpha)^{y+1}}{y!}.$$

   So:

$$\hat{\theta}_{MMSE}(y) = \frac{1}{y!}\int_0^\infty \theta^{y+1}e^{-(\alpha+1)\theta}d\theta(1+\alpha)^{y+1} = \frac{y+1}{\alpha+1};$$

$$\hat{\theta}_{MAP}(y) = \arg\max_{\theta>0}\left[y\log\theta - (\alpha+1)\theta\right] = \frac{y}{\alpha+1};$$

and $\hat{\theta}_{MMAE}(y)$ solves

$$\int_0^{\hat{\theta}_{MMAE}(y)} \pi_y(\theta)d\theta = \frac{1}{2}$$

which reduces to

$$\sum_{k=0}^{y} \frac{\left[\hat{\theta}_{MMAE}(y)\right]^k}{k!} = \frac{1}{2}e^{\hat{\theta}_{MMAE}(y)}.$$

Note that the series on the left-hand side is the truncated power series expansion of $\exp\{\hat{\theta}_{MMAE}(y)\}$ so that the $\hat{\theta}_{MMAE}(y)$ is the value at which this truncated series equals half of its untruncated value when the truncation point is y.

3. 5 points. Poor textbook Chapter IV, Problem 4.

   Solution:

   (a) MMSE estimate

   As usual, we need the posterior density on the parameters

$$\pi_y(\theta) = w(\theta|y) = \frac{p_\theta(y)w(\theta)}{p(y)}$$

First we calculate $p(y)$:

$$\begin{aligned} p(y) &= \int_\Lambda p_\theta(y)w(\theta)d\theta \\ &= \frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(y-1)^2\} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(y+1)^2\} \end{aligned}$$

Then we can compute the posterior density:

$$\begin{aligned} \pi_y(\theta) = w(\theta|y) &= \begin{cases} \frac{\frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(y-1)^2\}}{\frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(y-1)^2\}+\frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(y+1)^2\}} & \text{if } \theta = 1 \\ \frac{\frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(y+1)^2\}}{\frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(y-1)^2\}+\frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2\sigma^2}(y+1)^2\}} & \text{if } \theta = -1 \end{cases} \\ &= \begin{cases} \frac{\exp\{\frac{y}{\sigma^2}\}}{\exp\{\frac{y}{\sigma^2}\}+\exp\{-\frac{y}{\sigma^2}\}} & \text{if } \theta = 1 \\ \frac{\exp\{-\frac{y}{\sigma^2}\}}{\exp\{\frac{y}{\sigma^2}\}+\exp\{-\frac{y}{\sigma^2}\}} & \text{if } \theta = -1 \end{cases} \end{aligned}$$

The MMSE estimate is the conditional mean,

$$\begin{aligned} \hat{\theta}_{MMSE}(y) &= \int_\Lambda \theta\pi_y(\theta)d\theta \\ &= w(\theta=1|y) - w(\theta=-1|y) \\ &= \frac{\exp\frac{y}{\sigma^2} - \exp-\frac{y}{\sigma^2}}{\exp\frac{y}{\sigma^2} + \exp-\frac{y}{\sigma^2}} \\ &= \frac{2\sinh\left(\frac{y}{\sigma^2}\right)}{2\cosh\left(\frac{y}{\sigma^2}\right)} = \tanh\left(\frac{y}{\sigma^2}\right) \end{aligned}$$

(b) MAP estimate

$$\begin{aligned}
\hat{\theta}_{MAP}(y) &= \arg\max_{\theta=\pm 1} \pi_y(\theta) \\
&= \begin{cases} 1 & \text{if } \frac{\exp \frac{y}{\sigma^2}}{\exp \frac{y}{\sigma^2} + \exp -\frac{y}{\sigma^2}} \geq \frac{\exp -\frac{y}{\sigma^2}}{\exp \frac{y}{\sigma^2} + \exp -\frac{y}{\sigma^2}} \\ -1 & \text{if } \frac{\exp \frac{y}{\sigma^2}}{\exp \frac{y}{\sigma^2} + \exp -\frac{y}{\sigma^2}} < \frac{\exp -\frac{y}{\sigma^2}}{\exp \frac{y}{\sigma^2} + \exp -\frac{y}{\sigma^2}} \end{cases} \\
&= \begin{cases} 1 & \text{if } y \geq 0 \\ -1 & \text{if } y < 0 \end{cases} \\
&= \operatorname{sgn}(y)
\end{aligned}$$

(c) We can see that $\tanh(x) \to 1$ as $x \to \infty$ and similarly, $\tanh(x) \to -1$ as $x \to -\infty$. Thus, if σ^2 is very small ($\ll 1$), then the two estimators will be approximately equal.

4. 5 points. Poor textbook Chapter IV, Problem 7.

Solution:

We have

$$p_\theta(y) = \begin{cases} e^{-y+\theta} & \text{if } y \geq \theta \\ 0 & \text{if } y < \theta \end{cases} = e^{-y+\theta} I_{[\theta,\infty)}(y) = e^{-y+\theta} I_{[-\infty,y)}(\theta)$$

and

$$w(\theta) = \begin{cases} 1 & \text{if } \theta \in (0,1) \\ 0 & \text{if } \theta \notin (0,1) \end{cases} = I_{(0,1)}(\theta).$$

Thus,

$$\pi_y(\theta) = \frac{e^{-y+\theta}\mathbb{I}_{(0,\min\{1,y\})}(\theta)}{\int_0^{\min\{1,y\}} e^{-y+\theta} d\theta} = \frac{e^{\theta}\mathbb{I}_{(0,\min\{1,y\})}(\theta)}{e^{\min\{1,y\}} - 1}$$

where $\mathbb{I}_{a,b}(x)$ is the indicator function equal to one when $x \in (a,b)$ and equal to zero otherwise. This implies:

$$\hat{\theta}_{MMSE}(y) = \frac{\int_0^{\min\{1,y\}} \theta e^{\theta} d\theta}{e^{\min\{1,y\}} - 1} = \frac{[\min\{1,y\} - 1]e^{\min\{1,y\}} + 1}{e^{\min\{1,y\}} - 1};$$

$$\hat{\theta}_{MAP}(y) = \arg\left[\max_\theta e^{\theta} I_{(0,\min\{1,y\})}(\theta)\right] = \min\{1,y\};$$

and

$$\frac{\int_{-\infty}^{\hat{\theta}_{ABS}(y)} e^{\theta} I_{(0,\min\{1,y\})}(\theta)\ d\theta}{e^{\min\{1,y\}} - 1} = \frac{1}{2}$$

which has the solution

$$\hat{\theta}_{ABS}(y) = \log\left[\frac{e^{\min\{1,y\}} + 1}{2}\right].$$