# ECE531 Homework Assignment Number 7

Solution

Make sure your reasoning and work are clear to receive full credit for each problem.

1. 4 points. Poor textbook Chapter IV, Problem 12.

   **Solution:**

   The joint density of the $n$ observations parameterized by $\theta$ is,

$$
\begin{aligned}
p_Y(y\,;\,\theta) &= \begin{cases} (\theta-1)^n \prod_{k=1}^n y_k^{-\theta} & y_k \geq 1 \;\; \forall\, k \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} (\theta-1)^n \exp\left(-\theta \sum_{k=1}^n \ln y_k\right) & y_k \geq 1 \;\; \forall\, k \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

   Let

$$
\begin{aligned}
C(\theta) &= (\theta-1)^n \\
h(y) &= 1
\end{aligned}
$$

   From this factorization, it should be clear that $p_Y(y\,;\,\theta)$ is a single-parameter exponential family with $T(y) = -\sum_{k=1}^n \log y_k$. Using the Completeness Theorem for Exponential Families, we know that $T(y)$ is a complete sufficient statistic. Note that the negative sign doesn't matter here, i.e. $T(y) = \sum_{k=1}^n \log y_k$ is also a complete sufficient statistic.

2. 4 points. Suppose $Y_k \overset{\text{i.i.d.}}{\sim} p_Y(y\,;\,\theta)$ for $k = 0, \dots, n-1$ where

$$
p_Y(y\,;\,\theta) = \begin{cases} \frac{y}{\sigma^2} \exp\left(\frac{-y^2}{2\sigma^2}\right) & y > 0 \\ 0 & \text{otherwise.} \end{cases}
$$

   (a) Find a non-trivial sufficient statistic for estimating $\theta = \sigma^2$.

   (b) Is your sufficient statistic complete?

   **Solution:**

   (a) We have the joint density of the observations as a function of the unknown parameter $\theta$ as

$$
p_Y(y\,;\,\theta) = \begin{cases} \frac{1}{\theta} \left(\prod_{k=1}^n y_k\right) e^{-\frac{\sum_{k=1}^n (y_k^2)}{2\theta}} & \text{if } y_k > 0 \;\; \forall\, k \\ 0 & \text{otherwise.} \end{cases}
$$

   Let

$$
T(y) = \sum_{k=1}^n y_k^2,
$$

$$h(y) = \prod_{k=1}^{n} y_k,$$

$$g_\theta(T(y)) = \frac{1}{\theta} e^{-\frac{\sum_{k=1}^{n}(y_k^2)}{2\theta}}$$

From the Neyman-Fisher Factorization Theorem, we know that $T(y)$ is a sufficient statistic for the estimation of $\theta = \sigma^2$.

(b) From the supplementary material emailed on 26-March, it should be easy to see that $p_Y(y;\theta)$ is an exponential family with $d(\theta) = 1/\theta$. Hence, the Completeness Theorem for Exponential Families implies that $T(y)$ is a complete sufficient statistic.

3. 4 points. Repeat the previous problem for the case $Y_k \overset{\text{i.i.d.}}{\sim} \mathcal{U}[-\theta, \theta]$ for $k = 0, \dots, n-1$.

**Solution:**

(a) We have the joint density of the observations as a function of the unknown parameter $\theta$ as

$$p_Y(y;\theta) = \begin{cases} (\frac{1}{2\theta})^n & \text{if } -\theta \leq y \leq \theta \\ 0 & \text{otherwise} \end{cases}.$$

An intuitive choice for a sufficient statistic is

$$T(y) = \max(|y_k|),$$

which implies that

$$h(y) = \frac{1}{\max(|y_k|)},$$

and

$$g_\theta(T(y)) = \left(\frac{1}{2\theta}\right)^n T(y)$$

From Neyman-Fisher factorization theorem, this $T(y)$ is a sufficient statistic. There are, of course, other possibilities here.

(b) To check the completeness of $T(y)$, we can write the distribution of $Z = T(Y) = \max(|Y_k|)$ as

$$p_Z(z;\theta) = \frac{n}{\theta}\left(\frac{z}{\theta}\right)^{n-1}, \quad 0 \leq z \leq \theta$$

If $Z = T(Y)$ is not complete, then there will exist a non-zero function $f(Z)$ such that $\mathrm{E}_\theta[f(Z)] = 0$ for all $\theta \in \Lambda$, i.e.

$$\int_0^\theta f(z)\frac{n}{\theta}\left(\frac{z}{\theta}\right)^{n-1} dz = 0, \text{for all } \theta \in \Lambda$$

Note that $f(z)$ can't be non-zero on just a set of discrete points in $(0,\theta)$ because $\mathrm{Prob}[f(Z) = 0] = 1$ in this case. The function $f(z)$ must be non-zero (except for discrete points) on at least one subinterval $[\theta_0, \theta_1] \subset (0, \theta)$ with $\theta_1 > \theta_0$. Let $w(z) = f(z)nz^{n-1}$. By the definition of $w(z)$, this implies that $w(z)$ must be non-zero (except for discrete points) on this subinterval as well.

Using our definition for $w(z)$, we have

$$\frac{1}{\theta^n}\int_0^\theta w(z)dz = 0, \text{for all } \theta \in \Lambda$$

Suppose we find a non-zero $w(z)$ that causes $\int_0^\theta w(z)dz = 0$ for all $\theta \in \Lambda$. Then this must also be true for all $\theta \in [\theta_0, \theta_1]$. For these $\theta$, we can write

$$\int_0^\theta w(z)\,dz = \int_0^{\theta_0} w(z)\,dz + \int_{\theta_0}^\theta w(z)\,dz = 0 + \int_{\theta_0}^\theta w(z)\,dz = 0$$

But recall that $w(z)$ is nonzero on $[\theta_0, \theta_1]$. A function $w(z)$ that is nonzero on $[\theta_0, \theta_1]$ such that $\int_{\theta_0}^\theta W(z)\,dz = 0$ must have as much positive area as negative area on $[\theta_0, \theta]$. This is possible for any fixed $\theta$, but impossible for all $\theta \in [\theta_0, \theta_1]$. Hence $T(y) = \max(|y_k|)$ is complete.

4. 4 points. Recall the linear model $Y = H\theta + W$ discussed in lecture with observations $Y \in \mathbb{R}^n$. Assume that $W \sim \mathcal{N}(0, \sigma^2 I)$ with $\sigma^2$ known. Find the MVU estimator of $\theta \in \mathbb{R}^m$ by using the Neyman-Fisher factorization and the Rao-Blackwell-Lehmann-Sheffe theorems. Hint: This is a non-random parameter estimation problem. You do not have a prior distribution on $\theta$. Another hint: If you are having trouble with the vector case, try $m = 1$ and $n = 1$ first.

   **Solution:** The joint density can be written as

$$\begin{aligned} p_Y(y;\theta) &= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{(y-H\theta)^T(y-H\theta)}{2\sigma^2}} \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{y^T y - 2\theta^T H^T y - (H\theta)^T(H\theta)}{2\sigma^2}} \end{aligned}$$

   Let

$$h(y) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{\frac{y^T y}{2\sigma^2}}$$

$$\alpha(\theta) = e^{-\frac{(H\theta)^T H\theta}{2\sigma^2}}$$

   Note that $p_Y(y;\theta)$ belongs to exponential family and that

$$T(y) = H^T y.$$

   Hence $T(y)$ is sufficient and complete. We can find the MVU estimator by the RBLS theorem.

$$\tilde{g}[T(y)] = E_\theta[\hat{g}(Y)|T(Y) = T(y)]$$

   Note that $(H^T H)^{-1} H^T y$ is an unbiased estimator of $\theta$, hence let $\hat{g}(Y) = (H^T H)^{-1} H^T Y$ we have

$$\tilde{g}[T(y)] = E_\theta[(H^T H)^{-1} H^T Y | H^T Y = H^T y] = (H^T H)^{-1} H^T y.$$

   Hence $\hat{\theta}_{MVU}(y) = (H^T H)^{-1} H^T y$.

5. 4 points. Poor textbook Chapter IV, Problem 13(a).

   **Solution:** We can write the joint distribution parameterized by $\theta$ as

$$p_Y(y\,;\,\theta) = \theta^{T(y)}(1-\theta)^{(n-T(y))},$$

   where

$$T(y) = \sum_{k=1}^{n} y_k$$

   is the number of heads observed. From the Neyman-Fisher factorization theorem, it is easy to see that $T(y)$ is a sufficient statistic Note that $Z = T(Y)$ is binomially distributed with parameter $\theta$, and hence belongs to the one-parameter exponential family (see *Statistical Theory*, 4th edition, Bernard W. Lindgren, pp. 188-189). Hence $T(y)$ is complete. Finally, from the RBLS theorem we can compute the MVU estimator of $\theta$ as

$$\hat{\theta}_{MVU}(y) = \frac{1}{n}\sum_{k=1}^{n} y_k.$$