ECE531 Homework Assignment Number 9

Solution

Make sure your reasoning and work are clear to receive full credit for each problem.

1. 4 points. Poor textbook Chapter IV, Problem 13 (b).

   Solution:

   (b) The maximum likelihood estimator can be computed as

   $$\hat{\theta}_{\mathrm{ML}}(y) = \arg\max_{0<\theta<1} \left\{ \theta^{T(y)}(1-\theta)^{(n-T(y))} \right\}$$

   where $T(y) = \sum_{k=1}^{n} y_k$. Taking the logarithm and the partial derivative with respect to θ, and then setting the result equal to zero yields

   $$\frac{T(y)}{\theta} - \frac{n - T(y)}{1-\theta} = 0$$

   which can be rearranged to yield the desired result

   $$\hat{\theta}_{\mathrm{ML}}(y) = \frac{T(y)}{n}$$

   Note that $\hat{\theta}_{\mathrm{ML}}(y) = \hat{\theta}_{\mathrm{MVU}}(y)$ (see the solution to part (a)). Since the ML estimator is equal to the MVU estimator, we have immediately that $\mathrm{E}_\theta\{\hat{\theta}_{\mathrm{ML}}(Y)\} = \theta$. The variance of $\hat{\theta}_{\mathrm{ML}}(Y)$ is also easily computed to be $\mathrm{var}_\theta\{\hat{\theta}_{\mathrm{ML}}(Y)\} = \theta(1-\theta)/n$.

2. 4 points. Poor textbook Chapter IV, Problem 20 (c) and (d).

   Solution:

   (c) With $s_k^2 = 1$, the likelihood equation yields the solution

   $$\hat{\theta}(y) = \left(\frac{1}{n} \sum_{k=1}^{n} y_k^2 \right) - 1,$$

   which can be verified to yield a maximum of the likelihood function.

   (d) We have

   $$\mathrm{E}_\theta \left\{ \hat{\theta}_{\mathrm{ML}}(Y) \right\} = \left(\frac{1}{n} \sum_{k=1}^{n} E_\theta \left\{ Y_k^2 \right\} \right) - 1 = \theta.$$

   Similarly, since the $Y_k's$ are independent,

   $$\mathrm{var}_\theta \left(\hat{\theta}_{\mathrm{ML}}(Y) \right) = \frac{1}{n^2} \sum_{k=1}^{n} \mathrm{var}_\theta \left(Y_k^2 \right) = \frac{1}{n^2} \sum_{k=1}^{n} 2(1+\theta)^2 = \frac{2(1+\theta)^2}{n}.$$

   Thus, the ML estimator is unbiased and the variance of the ML estimator equals the CRLB. This implies that the ML estimator is equivalent to the MVU estimator in this case.

3. 4 points. Poor textbook Chapter IV, Problem 21 (d).

Solution: Recall that

$$p_Y(y\,;\,\lambda) \;=\; p_{Y_1}(y_1\,;\,\lambda)p_{Y_2}(y_2\,;\,\lambda) \;=\; \frac{\lambda^{(y_1+y_2)}e^{-2\lambda}}{y_1!y_2!}\mathbb{I}_{\{y_1\geq 0,\; y_2\geq 0\}}$$

and that $\theta = e^{-\lambda}$.

(d) To find the maximum likelihood estimator of θ, we can compute

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left(\ln p_Y(y\,;\,\lambda)\right) &= \frac{\partial}{\partial\theta}\left((y_1+y_2)\ln\lambda - 2\lambda - \ln(y_1!\,y_2!)\right) \\
&= \frac{y_1+y_2}{\lambda}\frac{\partial\lambda}{\partial\theta} - 2\frac{\partial\lambda}{\partial\theta} \\
\frac{\partial\lambda}{\partial\theta} &= -\frac{1}{\theta}
\end{aligned}$$

Setting the derivative equal to zero,

$$\begin{aligned}
\left[\frac{y_1+y_2}{-\ln\theta} - 2\right]\left(\frac{-1}{\theta}\right)\bigg|_{\theta=\hat{\theta}_{\mathrm{ML}}} &= 0 \\
\Rightarrow\; \hat{\theta}_{\mathrm{ML}}(y) &= e^{-\frac{y_1+y_2}{2}} \\
\mathbf{E}_\lambda\left\{\hat{\theta}_{\mathrm{ML}}(Y)\right\} &= \mathbf{E}_\lambda\left\{e^{-\frac{T}{2}}\right\} \text{ for } T\sim \text{ poisson}(2\lambda) \\
&= \sum_{t=0}^{\infty} e^{-\frac{t}{2}}\frac{(2\lambda)^t e^{-2\lambda}}{t!} = e^{-2\lambda}\sum_{t=0}^{\infty}\frac{(2\lambda/\sqrt{e})^t}{t!} = e^{-2\lambda}e^{\frac{2\lambda}{\sqrt{e}}} \\
&= e^{-\lambda\cdot 2\left(1-\frac{1}{\sqrt{e}}\right)} \;\; \textbf{biased} \\
\text{note that } 2\left(1-\frac{1}{\sqrt{e}}\right) &\approx 0.7869
\end{aligned}$$

If we create the MLE for λ directly, however, we get

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\ln p_Y(y\,;\,\lambda) &= \frac{y_1+y_2}{\lambda} - 2 \\
\Rightarrow\; \hat{\lambda}_{\mathrm{ML}}(y) &= \frac{y_1+y_2}{2} \\
\mathbf{E}_\lambda\left\{\hat{\lambda}_{\mathrm{ML}}(Y)\right\} &= \lambda \;\; \textbf{unbiased}.
\end{aligned}$$

4. 4 points. Poor textbook Chapter IV, Problem 23.

Solution:

(a) The log-likelihood is

$$\log p_Y(y\,;\,A,\phi) \;=\; -\frac{1}{2\sigma^2}\sum_{k=1}^{n}\left[y_k - A\sin(\frac{k\pi}{2}+\phi)\right]^2 - \frac{n}{2}\log(2\pi\sigma^2)$$

and, differentiating with respect to the unknown parameters A and ϕ, we obtain the likelihood equations

$$\begin{aligned}
\sum_{k=1}^{n}\left[y_k - \hat{A}\sin(\frac{k\pi}{2}+\hat{\phi})\right]\sin(\frac{k\pi}{2}+\hat{\phi}) &= 0 \\
\hat{A}\sum_{k=1}^{n}\left[y_k - \hat{A}\sin(\frac{k\pi}{2}+\hat{\phi})\right]\cos(\frac{k\pi}{2}+\hat{\phi}) &= 0.
\end{aligned}$$

Using the identities $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, and defining the quantities

$$\begin{aligned} y_s &= \frac{2}{n}\sum_{k=1}^{n} y_k \sin(\frac{k\pi}{2}) \\ y_c &= \frac{2}{n}\sum_{k=1}^{n} y_k \cos(\frac{k\pi}{2}), \end{aligned}$$

we can rewrite the likelihood equations as the pair

$$\begin{aligned} \hat{A} &= y_s\cos(\hat{\phi}) + y_c\sin(\hat{\phi}) \\ 0 &= y_c\cos(\hat{\phi}) - y_s\sin(\hat{\phi}) \end{aligned}$$

For this last step we took advantage of the facts that, for even n,

$$\begin{aligned} \frac{n}{2} &= \sum_{k=1}^{n}\sin^2(\frac{k\pi}{2}) = \sum_{k=1}^{n}\cos^2(\frac{k\pi}{2}) \\ 0 &= \sum_{k=1}^{n}\sin(\frac{k\pi}{2})\cos(\frac{k\pi}{2}) \end{aligned}$$

Putting the likelihood equations together we find

$$\begin{aligned} \hat{A}^2 &= y_s^2\cos^2(\hat{\phi}) + y_c^2\sin^2(\hat{\phi}) + 2y_sy_c\cos(\hat{\phi})\sin(\hat{\phi}) \\ &= y_s^2\cos^2(\hat{\phi}) + y_c^2\sin^2(\hat{\phi}) + y_s^2\sin^2(\hat{\phi}) + y_c^2\cos^2(\hat{\phi}) \\ &= y_s^2 + y_c^2. \end{aligned}$$

Thus,

$$\begin{aligned} \hat{A} &= \sqrt{y_s^2 + y_c^2} \\ \hat{\phi} &= \tan^{-1}(\frac{y_c}{y_s}) \end{aligned}$$

(b) The joint MAP estimator of $[A, \phi]$ solves

$$\begin{aligned} \{\hat{A}_{\mathrm{MAP}}, \hat{\phi}_{\mathrm{MAP}}\} &= \arg\max_{a,\phi} w(a,\phi|y) \\ &= \arg\max_{a,\phi} \log w(a,\phi|y) \\ &= \arg\max_{a,\phi} \log \frac{p(y|a,\phi)w_A(a)w_\Phi(\phi)}{p(y)} \\ &= \arg\max_{a,\phi} \log p(y|a,\phi) + \log w_A(a) + \log w_\Phi(\phi) \\ &= \arg\max_{a,\ \phi\in[-\pi,\pi)} \log p(y|a,\phi) + \log w_A(a) \end{aligned}$$

We now search for the maximum of $\log p(y|a,\phi) + \log w_A(a)$ by setting the gradient with respect to $[a,\phi]$ to zero. Similar to before, we get

$$\begin{aligned} \sum_{k=1}^{n}\left[y_k - \hat{A}\sin(\frac{k\pi}{2}+\hat{\phi})\right]\sin(\frac{k\pi}{2}+\hat{\phi}) + \frac{\sigma^2}{\hat{A}} - \frac{\hat{A}\sigma^2}{\beta^2} &= 0 \\ \hat{A}\sum_{k=1}^{n}\left[y_k - \hat{A}\sin(\frac{k\pi}{2}+\hat{\phi})\right]\cos(\frac{k\pi}{2}+\hat{\phi}) &= 0. \end{aligned}$$

Using the previously mentioned trig identities and definitions of y_s and y_c, we can obtain the pair of equations

$$\begin{aligned} \hat{A}(1+\alpha) - \frac{2\sigma^2}{n\hat{A}} &= y_s \cos(\hat{\phi}) + y_c \sin(\hat{\phi}) \\ 0 &= y_c \cos(\hat{\phi}) - y_s \sin(\hat{\phi}) \end{aligned}$$

for $\alpha = \frac{2\sigma^2}{n\beta^2}$. Putting the previous equations together (as before) we find that

$$\hat{A}(1+\alpha) - \frac{2\sigma^2}{n\hat{A}} = \underbrace{\sqrt{y_s^2 + y_c^2}}_{\hat{A}_{\rm ML}}$$

and a simple application of the quadratic equation yields

$$\hat{A}_{\rm MAP} = \frac{\hat{A}_{\rm ML} + \sqrt{\hat{A}_{\rm ML}^2 + \frac{8\sigma^2(1+\alpha)}{n}}}{2(1+\alpha)}$$

It can be seen quite easily that $\hat{\phi}_{\rm MAP} = \hat{\phi}_{\rm ML}$.

(c) Note that, when $\beta \to \infty$, the MAP estimate of A does not approach the ML estimate of A. However, as $n \to \infty$, the MAP estimate does approach the ML estimate.

5. 4 points. Write a Matlab program to simulate one dimensional motion as described on Slide 14 of Lecture 10b, with $T = 1$ and $X[0] = [0, 0]^\top$. Your input sequence $U[n]$ should be Gaussian with zero mean and unit variance and autocorrelation

$$\mathrm{E}[U[n]U[k]] = \begin{cases} 1 & n = k \\ 0 & \text{otherwise.} \end{cases}$$

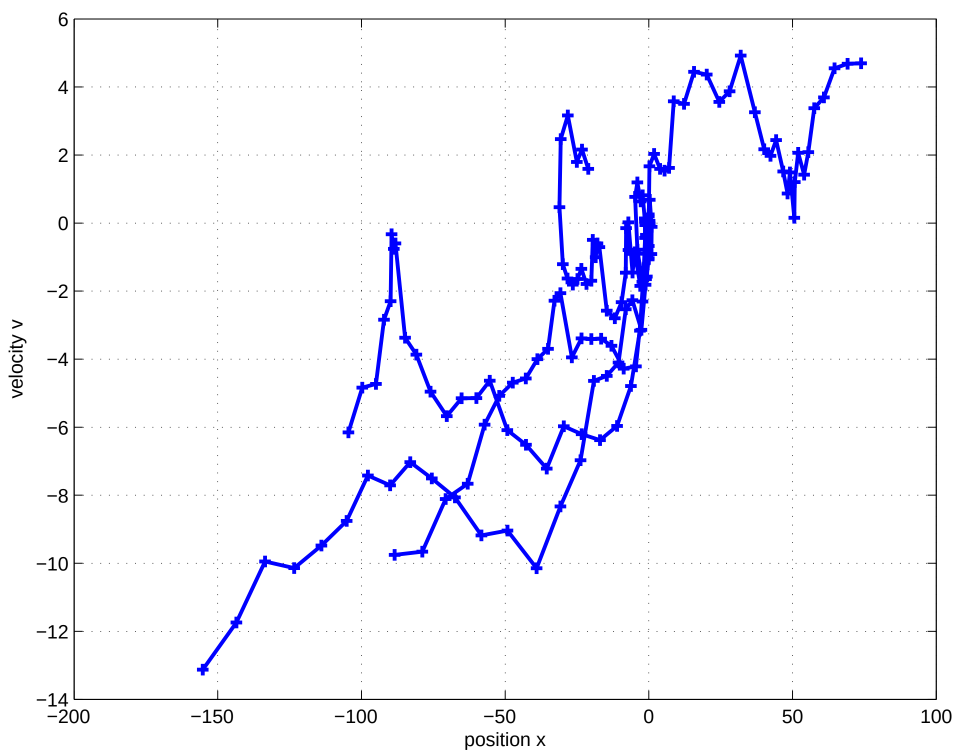
Plot the position and velocity of the particle (see slide 6 of Lecture 10b) for several different realizations of the input sequence. Also, assuming that your observations are noiseless, write Matlab code to generate estimates of the state $X[n-1]$ for $n \geq 1$ given observations $Y[0], Y[1], \ldots, Y[n]$ (this is a "smoothing" problem). You don't need to find an optimum estimator here; just use your intuition and confirm that your estimates are at least close to the true state.

Solution:

```
% ECE531 Spring 2009
% Solution to HW9 Problem 5

% ---------------------------------------------
% USER PARAMETERS
% ---------------------------------------------
iter = 5;               % number of iterations
N = 30;                         % number of steps in each iteration
T = 1;                          % discrete time step size
% ---------------------------------------------

X = zeros(2,N+1);          % declare state vector
G = [0 ; T];               % declare G matrix
F = [1 T ; 0 1];           % declare F matrix

for k=1:iter,

```

```
    % generate iid random Gaussian input
    U = randn(1,N);

    % generate states according to dynamic update equation
    for n=1:N,
        X(:,n+1) = F*X(:,n) + G*U(:,n);
    end

    % make plots
    plot(X(1,:),X(2,:),'-+','Linewidth',2);
    hold on

end

hold off
xlabel('position x');
ylabel('velocity v');
grid on
```

Here is a plot of 5 realizations of the one-dimensional motion simulator:



An intuitively "good" estimator of the unknown state might be

$$\hat{X}[n-1] \quad = \quad \begin{bmatrix} Y[n-1] \\ (Y[n]-Y[n-1])/T \end{bmatrix}.$$

The first element of the state (the position of the particle) can be estimated without error because we have noiseless observations of the position of the particle. The second element of the state (the velocity of the particle) can be estimated by looking at the change in position between time n and time $n-1$ and dividing by the elapsed time. This estimate will not be perfect because the input affects the position (not just the state), but this should be a "good" estimator, at least intuitively.