ECE531 Spring 2009 Midterm Examination

Instructions: This exam is worth a total of 340 points. You may consult one double-sized letter-sized sheet of notes (in your own handwriting) and you may use a calculator during the exam. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution. The exam is closed-book.

1. 120 points total. Suppose you work in a widget manufacturing facility and that, before shipping a widget to a customer, each widget must be tested on the Widget Tester to determine if the widget is "good" or "defective". The Widget Tester has three lights:

   - red (definitely defective),
   - yellow (might be defective), and
   - green (definitely not defective).

   When a "good" widget is tested, the green light turns on with probability p and the yellow light turns on with probability $1-p$. When a "defective" widget is tested, the red light turns on with probability p and the yellow light turns on with probability $1-p$.

   (a) 10 points. What kind of hypothesis testing problems is this?

   (b) 30 points. Set up this hypothesis testing problem by explicitly defining the states, hypotheses, observations, and the conditional distributions on the observations for each state.

   (c) 10 points. How many deterministic decision rules are there?

   (d) 50 points. Suppose the cost of throwing away a good widget is \$100 and the cost of shipping a bad widget to the customer is \$100. For the case $p = 0.5$, sketch the Pareto-optimal risk tradeoff surface as accurately as possible. Label all vertices.

   (e) 20 points. Find a decision rule D^* that gives conditional risks $R_0(D^*) = R_1(D^*) = \$25$.

2. 120 points total. Suppose you observe one random variable $Y \in \mathbb{R}$ and must decide between

$$\begin{aligned} \mathcal{H}_0 &: Y \sim \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y^2}{2}\right) \\ \mathcal{H}_1 &: Y \sim \frac{1}{2\sqrt{2\pi}} \exp\left(\frac{-y^2}{8}\right) \end{aligned}$$

   (a) 40 points. Assume the uniform cost assignment and find a Bayes decision rule for a general prior.

   (b) 40 points. Find the least favorable prior and the minimax decision rule for this problem (also assume the UCA). Hint: You can use the approximation $Q(x) \approx \frac{1}{2} - \frac{x}{\sqrt{2\pi}}$ here. This approximation results from a Taylor expansion of $Q(x)$ around $x = 0$.

(c) 40 points. Find a Neyman-Pearson decision rule with false positive probability α for this problem. Your answer can include one or more Q^{-1} functions.

In all cases, please describe the threshold(s) of your decision rule as explicitly as you can.

3. 100 points total. Suppose you observe one random variable $Y \in [0, \infty)$ and must decide between

$$\begin{aligned} \mathcal{H}_0 &: Y \sim \exp(-y) \\ \mathcal{H}_1 &: Y \sim \lambda \exp(-\lambda y) \text{ for } 1 < \lambda < \infty \end{aligned}$$

where λ is unknown.

(a) 70 points total. Derive a UMP decision rule of size α if it exists, otherwise derive an LMP decision rule of size α. Note that, if the UMP decision rule exists, you do not need to find the LMP decision rule.

(b) 30 points. Derive and sketch the power function of your decision rule.