Some More Details on Slide 13 of ECE531 Lecture 5

D.R. Brown III

February 20, 2009

We want to find the index corresponding to the minimum of three discriminant functions:

$$\begin{aligned} g_0(y,\pi) &= \int_{\mathcal{X}} C_0(x)\pi(x)p_x(y)\,dx \\ g_1(y,\pi) &= \int_{\mathcal{X}} C_1(x)\pi(x)p_x(y)\,dx \\ g_2(y,\pi) &= \int_{\mathcal{X}} C_2(x)\pi(x)p_x(y)\,dx \end{aligned}$$

where $\mathcal{X} = [0,3)$, $\pi(x) = 1/3$,

$$\begin{aligned} C_0(x) &= \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < 3 \end{cases} \\ C_1(x) &= \begin{cases} 1 & 0 \le x < 1 \\ 0 & 1 \le x < 2 \\ 1 & 2 \le x < 3 \end{cases} \\ C_2(x) &= \begin{cases} 1 & 0 \le x < 2 \\ 0 & 2 \le x < 3 \end{cases} \end{aligned}$$

and

$$p_x(y) = \begin{cases} \frac{1}{4} & x-1 \le y_0 \le x+1 \text{ and } x-1 \le y_1 \le x+1 \\ 0 & \text{otherwise.} \end{cases}$$

Given all of this, we can rewrite the expressions for the commodity costs g_i as

$$\begin{aligned} g_0(y,\pi) &= \frac{1}{3}\int_1^3 p_x(y)\,dx \\ g_1(y,\pi) &= \frac{1}{3}\int_0^1 p_x(y)\,dx + \frac{1}{3}\int_2^3 p_x(y)\,dx \\ g_2(y,\pi) &= \frac{1}{3}\int_0^2 p_x(y)\,dx. \end{aligned}$$

Let's define

$$w(y) \quad := \quad \frac{1}{3}\int_0^3 p_x(y)\,dx$$

and $f_i(y,\pi) := w(y) - g_i(y,\pi)$ for $i \in \{0,1,2\}$. Note that $w(y)$ is not a function of x or i, i.e. given the observation y, $w(y)$ is just a constant. Hence,

$$\arg\min_{i\in\{0,1,2\}} g_i(y,\pi) \quad \Leftrightarrow \quad \arg\max_{i\in\{0,1,2\}} f_i(y,\pi).$$

We can explicitly write out the $f_i(y,\pi)$ terms as

$$\begin{aligned}
f_0(y,\pi) &= \frac{1}{3}\int_0^3 p_x(y) - \frac{1}{3}\int_1^3 p_x(y)\,dx = \frac{1}{3}\int_0^1 p_x(y)\,dx \\
f_1(y,\pi) &= \frac{1}{3}\int_0^3 p_x(y) - \frac{1}{3}\left(\int_0^1 p_y(x)\,dx + \int_2^3 p_y(x)\,dx\right) = \frac{1}{3}\int_1^2 p_x(y)\,dx \\
f_2(y,\pi) &= \frac{1}{3}\int_0^3 p_x(y) - \frac{1}{3}\int_0^2 p_y(x)\,dx = \frac{1}{3}\int_2^3 p_x(y)\,dx.
\end{aligned}$$

Note that the $\frac{1}{3}$ factor common to each $f_i(y,\pi)$ is irrelevant to the maximization. Hence, defining $h_i(y,\pi) := 3f_i(y,\pi)$ for $i \in \{0,1,2\}$, we can say that

$$\arg\min_{i\in\{0,1,2\}} g_i(y,\pi) \quad \Leftrightarrow \quad \arg\max_{i\in\{0,1,2\}} f_i(y,\pi) \quad \Leftrightarrow \quad \arg\max_{i\in\{0,1,2\}} h_i(y,\pi).$$

with

$$\begin{aligned}
h_0(y,\pi) &= \int_0^1 p_x(y)\,dx \\
h_1(y,\pi) &= \int_1^2 p_x(y)\,dx \\
h_2(y,\pi) &= \int_2^3 p_x(y)\,dx.
\end{aligned}$$

Hopefully this makes the example a bit more clear. As a final (somewhat unrelated note), A step that I didn't really explain in lecture, but might be helpful in actually computing the integrals, is that we can rewrite the conditional pdfs as

$$p_x(y) \quad = \quad \begin{cases} \frac{1}{4} & \max\{y_0, y_1\} - 1 \le x \le \min\{y_0, y_1\} + 1 \\ 0 & \text{otherwise.} \end{cases}$$

As an example, suppose we get a vector observation $y = [y_0, y_1] = [0, 1]$. The first observation tells us that $-1 \le x \le 1$ and the second observation tells us that $0 \le x \le 2$. The state must lie in the intersection of these two intervals, hence $0 \le x \le 1$. You can derive the same expression that I have above from this intuition.