ECE531 Lecture 3c: Bayesian Hypothesis Testing

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Minimizing a Weighted Sum of Conditional Risks

We would like to minimize all of the conditional risks $R_0(D), \dots, R_{N-1}(D)$, but we have to make some sort of tradeoff. A standard approach to solving a **multi-objective optimization problem** is to form a **linear combination** of the functions to be minimized:

$$r(D,\lambda) := \sum_{j=0}^{N-1} \lambda_j R_j(D) = \lambda^\top R(D)$$

where $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$.

The problem is then to solve the **single-objective optimization problem**:

$$\begin{aligned} D^* &= \arg\min_{D \in \mathcal{D}} r(D,\lambda) \\ &= \arg\min_{D \in \mathcal{D}} \sum_{j=0}^{N-1} \lambda_j c_j^\top D p_j \end{aligned}$$

where we are focusing on the case of finite $\mathcal{Y}$ for now.

Minimizing a Weighted Sum of Conditional Risks

The problem

$$D^* = \arg\min_{D \in \mathcal{D}} r(D,\lambda) = \arg\min_{D \in \mathcal{D}} \sum_{j=0}^{N-1} \lambda_j c_j^\top D p_j$$

is a **finite-dimensional linear programming** problem since the variables that we control (D_{ij}) are constrained by linear inequalities

$$\begin{aligned} D_{ij} &\geq 0 \;\; \forall i \in \{0,\dots,M-1\} \text{ and } j \in \{0,\dots,L-1\} \\ \sum_{i=0}^{M-1} D_{ij} &= 1 \end{aligned}$$

and the objective function $r(D,\lambda)$ is linear in the variables D_{ij}. For notational convenience, we call this minimization problem LPF(λ).

Decomposition: Part 1

Since

$$R_j(D) = c_j^\top D p_j = \sum_{i=0}^{M-1} C_{ij} \sum_{\ell=0}^{L-1} D_{i\ell} P_{\ell j}$$

the problem LPF(λ) can be written as

$$\begin{aligned} D^* &= \arg\min_{D \in \mathcal{D}} \sum_{j=0}^{N-1} \lambda_j \sum_{i=0}^{M-1} C_{ij} \sum_{\ell=0}^{L-1} D_{i\ell} P_{\ell j} \\ &= \arg\min_{D \in \mathcal{D}} \sum_{\ell=0}^{L-1} \underbrace{\left(\sum_{i=0}^{M-1} D_{i\ell} \left[\sum_{j=0}^{N-1} \lambda_j C_{ij} P_{\ell j} \right] \right)}_{r_\ell(d,\lambda)} \end{aligned}$$

Note that we can minimize each term $r_\ell(d, \lambda)$ separately.

Decomposition: Part 2

The ℓth subproblem is then

$$\begin{aligned} d^* &= \arg\min_{d \in \mathbb{R}^M} r_\ell(d, \lambda) \\ &= \arg\min_{d} \sum_{i=0}^{M-1} d_i \left[\sum_{j=0}^{N-1} \lambda_j C_{ij} P_{\ell j} \right] \end{aligned}$$

subject to the constraints that $d_i \geq 0$ and $\sum_i d_i = 1$.

- Note that $d_i \leftrightarrow D_{i\ell}$ from the original LPF(λ) problem.
- For notational convenience, we call this subproblem LPF$_\ell(\lambda)$.

Some Intuition: The Hiker

Suppose you are going on a hike with a one liter drinking bottle and you must completely fill the bottle before your hike. At the general store, you can fill your bottle with any combination of:

- Tap water: $0.25 per liter
- Ice tea: $0.50 per liter
- Soda: $1.00 per liter
- Red Bull: $5.00 per liter

You want to minimize your cost for the hike. What should you do?
We can define a per-liter cost vector

$$h = [0.25, 0.50, 1.00, 5.00]^\top$$

and a proportional purchase vector $\nu \in \mathbb{R}^4$ with elements satisfying $\nu_i \geq 0$ and $\sum_i \nu_i = 1$. Clearly, your cost $h^\top \nu$ is minimized when $\nu = [1, 0, 0, 0]^\top$.

Solution to the Subproblem $\mathsf{LPF}_\ell(\lambda)$

Theorem

For fixed $h \in \mathbb{R}^n$, the function $h^\top \nu$ is minimized over the set $\nu \in \{\mathbb{R}^n : \nu_i \geq 0$ and $\sum_i \nu_i = 1\}$ by the vector ν^* with $\nu^*_m = 1$ and $\nu^*_k = 0$, $k \neq m$, where m is any index with $h_m = \min_{i \in \{0,\dots,n-1\}} h_i$.

Note that, if two or more commodities have the same minimum price, the solution is not unique. Back to our subproblem $\mathsf{LPF}_\ell(\lambda)$ ...

$$d^* = \arg \min_{d \in \mathbb{R}^M} \sum_{i=0}^{M-1} d_i \left[\sum_{j=0}^{N-1} \lambda_j C_{ij} P_{\ell j} \right]$$

subject to the constraints that $d_i \geq 0$ and $\sum_i d_i = 1$. The Theorem tells us how to easily solve $\mathsf{LPF}_\ell(\lambda)$. We just have to find the index

$$m_\ell = \arg \min_{i \in \{0,\dots,M-1\}} \left[\sum_{j=0}^{N-1} \lambda_j C_{ij} P_{\ell j} \right]$$

and then set $d_{m_\ell} = 1$ and $d_i = 0$ for all $i \neq m_\ell$.

Solution to the Problem LPF(λ)

Solving the subproblem LPF$_\ell(\lambda)$ for $\ell = 0, \dots, L-1$ and assembling the results of the subproblems into $M \times L$ decision matrices yields a non-empty, finite set of **deterministic** decision matrices that solve LPF(λ).

All deterministic decision matrices in the set have the same minimal cost:

$$r^*(\lambda) \;=\; \sum_{\ell=0}^{L-1} \min_i \left\{ \sum_{j=0}^{N-1} \lambda_j C_{ij} P_{\ell j} \right\}$$

It is easy to show that any randomized decision rule in the convex hull of this set of deterministic decision matrices also achieves $r^*(\lambda)$, and hence is also optimal.

Example Part 1

Let $G_{i\ell} := \sum_{j=0}^{N-1} \lambda_j C_{ij} P_{\ell j}$ and $G \in \mathbb{R}^{M \times L}$ be the matrix composed of elements $G_{i\ell}$. Suppose

$$G = \begin{bmatrix} 0.3 & 0.5 & 0.2 & 0.8 \\ 0.4 & 0.2 & 0.1 & 0.5 \\ 0.5 & 0.1 & 0.7 & 0.6 \end{bmatrix}$$

- What is the solution to the subproblem $\mathsf{LPF}_0(\lambda)$? $d = [1, 0, 0]^\top$.
- What is the solution to the problem $\mathsf{LPF}(\lambda)$?

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- Is this solution unique? Yes.
- What is $r^*(\lambda)$? $r^*(\lambda) = 1.0$.

Example Part 2

- What happens if

$$G = \begin{bmatrix} 0.3 & 0.5 & 0.2 & 0.8 \\ 0.4 & 0.2 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.7 & 0.6 \end{bmatrix}?$$

The solution to $\text{LPF}_2(\lambda)$ is no longer unique. Both

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ or } D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

solve $\text{LPF}(\lambda)$ and achieve $r^*(\lambda) = 1.1$.

- In fact, any decision matrix of the form

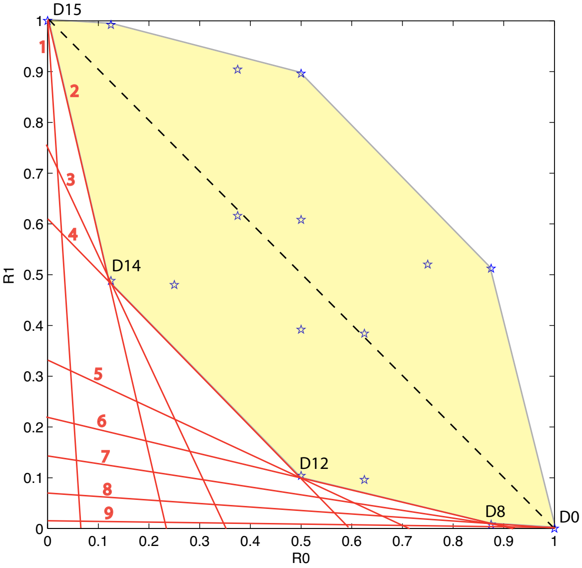
$$D = \begin{bmatrix} 1 & 0 & \alpha & 0 \\ 0 & 0 & 1-\alpha & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

for $\alpha \in [0, 1]$ will also achieve $r^*(\lambda) = 1.1$.

Remarks

- Note that the solution to the subproblem $\text{LPF}_\ell(\lambda)$ always yields at least one deterministic decision rule for the case $y = y_\ell$.
- If n_ℓ elements of the column G_ℓ are equal to the minimum of the column, then there are n_ℓ deterministic decision rules solving the subproblem $\text{LPF}_\ell(\lambda)$.
- There are $\prod_\ell n_\ell \geq 1$ deterministic decision rules that solve $\text{LPF}(\lambda)$.
- If there is more than one deterministic solution to $\text{LPF}(\lambda)$ then there will also be a corresponding convex hull of randomized decision rules that also achieve $r^*(\lambda)$.

Working Example: Effect of Changing the Weighting λ

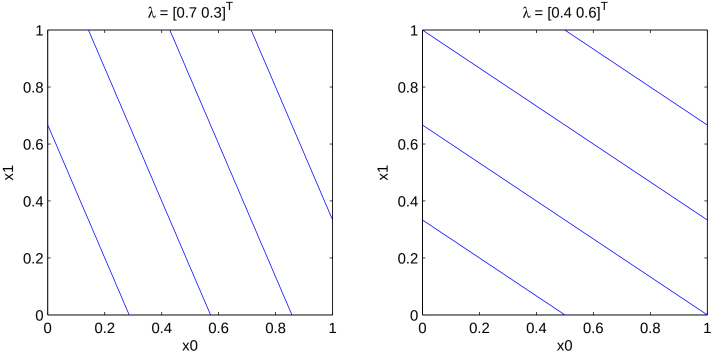


Weighted Risk Level Sets

Notice in the last example that, by varying our risk weighting λ, we can create a family of "level sets" that cover every point of the optimal tradeoff surface.

Given a constant $c \in \mathbb{R}$, the **level set** of value c is defined as

$$L_c^\lambda \quad := \quad \{x \in R^N : \lambda^\top x = c\}$$



Infinite Observation Spaces: Part 1

When the observations are specified by conditional pdfs, our conditional risk function becomes

$$R_j(\rho) \;=\; \int_{y\in\mathcal{Y}} \left[\sum_{i=0}^{M-1} \rho_i(y) C_{ij}\right] p_j(y)\, dy$$

Recall that the randomized decision rule $\rho_i(y)$ specifies the probability of deciding $\mathcal{H}_i$ when the observation is y.

We still have $N < \infty$ states, so, as before, we can write a linear combination of the conditional risk functions as

$$r(\rho,\lambda) := \sum_{j=0}^{N-1} \lambda_j R_j(\rho)$$

where $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$.

Infinite Observation Spaces: Part 2

The problem is then to solve

$$\begin{aligned}\rho^* &= \arg\min_{\rho\in\mathcal{D}} r(\rho,\lambda)\\ &= \arg\min_{\rho\in\mathcal{D}} \sum_{j=0}^{N-1} \lambda_j \int_{y\in\mathcal{Y}} \left[\sum_{i=0}^{M-1} \rho_i(y) C_{ij}\right] p_j(y)\,dy\end{aligned}$$

where $\mathcal{D}$ is the set of all valid decision rules satisfying $\rho_i(y) \geq 0$ for all $i = 0,\dots,M-1$ and $y \in \mathcal{Y}$, as well as $\sum_{i=0}^{M-1} \rho_i(y) = 1$ for all $y \in \mathcal{Y}$.

- This is an infinite dimensional linear programming problem.
- We refer to this problem as LP(λ).

Infinite Observation Spaces: Decomposition Part 1

As before, we can decompose the big problem LP(λ) into lots of little subproblems, one for each observation $y \in \mathcal{Y}$. We rewrite LP(λ) as

$$\begin{aligned}\rho^* &= \arg\min_{\rho \in \mathcal{D}} \sum_{j=0}^{N-1} \lambda_j \int_{y \in \mathcal{Y}} \left[\sum_{i=0}^{M-1} \rho_i(y) C_{ij}\right] p_j(y)\, dy \\ &= \arg\min_{\rho \in \mathcal{D}} \int_{y \in \mathcal{Y}} \left(\sum_{i=0}^{M-1} \rho_i(y) \left[\sum_{j=0}^{N-1} \lambda_j C_{ij} p_j(y)\right]\right) dy\end{aligned}$$

To minimize the integral, we can minimize the term

$$(\dots) = \left(\sum_{i=0}^{M-1} \rho_i(y) \left[\sum_{j=0}^{N-1} \lambda_j C_{ij} p_j(y)\right]\right)$$

at each fixed value of y.

Infinite Observation Spaces: Decomposition Part 2

The subproblem of minimizing the term $(\dots)$ inside the integral when $y = y_0$ is a finite-dimensional linear programming problem (since $\mathcal{X}$ is still finite):

$$\nu^* = \arg\min_{\nu} \sum_{i=0}^{M-1} \nu_i \left[\sum_{j=0}^{N-1} \lambda_j C_{ij} p_j(y_0) \right]$$

subject to $\nu_i \geq 0$ for all $i = 0, \dots, M-1$ and $\sum_{i=0}^{M-1} \nu_i = 1$.

We already know how to do this. We just find the index that has the lowest commodity cost

$$m_{y_0} = \arg \min_{i \in \{0,\dots,M-1\}} \sum_{j=0}^{N-1} \lambda_j C_{ij} p_j(y_0)$$

and set $\nu_{m_{y_0}} = 1$ and all other $\nu_i = 0$. Same technique as solving $\text{LPF}_\ell(\lambda)$.

Infinite Observation Spaces: Minimum Weighted Risk

As we saw with finite $\mathcal{Y}$, solving LP(λ) will yield at least one deterministic decision rule that achieves the minimum weighted risk

$$\begin{aligned} r^*(\lambda) &= \int_{y\in\mathcal{Y}} \min_{i\in\{0,\dots,M-1\}} \sum_{j=0}^{N-1} \lambda_j C_{ij} p_j(y)\, dy \\ &= \int_{y\in\mathcal{Y}} \min_{i\in\{0,\dots,M-1\}} g_i(y,\lambda)\, dy \\ &= \sum_{i=0}^{M-1} \int_{y\in\mathcal{Y}_i^*} g_i(y,\lambda)\, dy \end{aligned}$$

1. The existence of at least one deterministic decision rule solving LPF(λ) or LP(λ) implies the existence of an **optimal partition** of $\mathcal{Y}$.
2. Since the problems LPF(λ) and LP(λ) always yield at least one deterministic decision rule, another way to think of these problems is: "Find a partition of the observation space that minimizes the weighted risk".

Infinite Observation Spaces: Putting it All Together

A decision rule that solves LP(λ) is then

$$\rho_i^*(y) = \begin{cases} 1 & \text{if } i = \arg\min_{m \in \{0,\dots,M-1\}} g_m(y,\lambda) \\ 0 & \text{otherwise} \end{cases}$$

where $\rho^*(y) = [\rho_0^*(y), \dots, \rho_{M-1}^*(y)]^\top$.

Remarks:

1. The solution to LP(λ) may not be unique.
2. If there is more than one deterministic solution to LP(λ) then there will also be a corresponding convex hull of randomized decision rules that also achieve $r^*(\lambda)$.
3. Note that the standard convention for deterministic decision rules is $\delta : \mathcal{Y} \mapsto \mathcal{Z}$. An equivalent way of specifying a deterministic decision rule that solves LP(λ) is

$$\delta^*(y) = \arg \min_{i \in \{0,\dots,M-1\}} g_i(y,\lambda).$$

The Bayesian Approach

We assume a prior state distribution $\pi \in \mathcal{P}_N$ such that

$$\mathrm{Prob}(\text{state is } x_j) \quad = \quad \pi_j$$

Note that this is our model of the state probabilities **prior to the observation**.

We denote the Bayes Risk of the decision rule ρ as

$$r(\rho, \pi) = \sum_{j=0}^{N-1} \pi_j R_j(\rho) = \sum_{j=0}^{N-1} \pi_j \int_{y \in \mathcal{Y}} p_j(y) \sum_{i=0}^{M-1} C_{ij} \rho_i(y)\, dy.$$

This is simply the weighted overall risk, or average risk, given our prior belief of the state probabilities. A decision rule that minimizes this risk is called a Bayes decision rule for the prior π.

Solving Bayesian Hypothesis Testing Problems

We know how to solve this problem. For finite $\mathcal{Y}$, we just find the index

$$m_\ell = \arg \min_{i \in \{0,\dots,M-1\}} \underbrace{\sum_{j=0}^{N-1} \pi_j C_{ij} P_{\ell j}}_{G_{i\ell}}$$

and set $D^{B\pi}_{m_\ell,\ell} = 1$ and $D^{B\pi}_{i,\ell} = 0$ for all $i \neq m_\ell$.
For infinite $\mathcal{Y}$, we just find the index

$$m_{y_0} = \arg \min_{i \in \{0,\dots,M-1\}} \underbrace{\sum_{j=0}^{N-1} \pi_j C_{ij} p_j(y_0)}_{g_i(y_0,\pi)}$$

and set $\delta^{B\pi}(y_0) = m_{y_0}$ (or, equivalently, set $\rho^{B\pi}_{m_{y_0}}(y_0) = 1$ and $\rho^{B\pi}_i(y_0) = 0$ for all $i \neq m_{y_0}$).

Prior and Posterior Probabilities

The conditional probability that $\mathcal{H}_j$ is true given the observation y:

$$\pi_j(y) := \mathrm{Prob}(\mathcal{H}_j \text{ is true} \,|\, Y = y) = \frac{p_j(y)\pi_j}{p(y)}$$

where

$$p(y) \;=\; \sum_{j=0}^{N-1} \pi_j p_j(y).$$

- Recall that π_j is the **prior** probability of choosing $\mathcal{H}_j$, before we have any observations.
- The quantity $\pi_j(y)$ is the **posterior** probability of choosing $\mathcal{H}_j$, conditioned on the observation y.

A Bayes Decision Rule Minimizes The Posterior Cost

General expression for a deterministic Bayes decision rule:

$$\delta^{B\pi}(y) = \arg \min_{i \in \{0,\dots,M-1\}} \sum_{j=0}^{N-1} \pi_j C_{ij} p_j(y)$$

But $\pi_j p_j(y) = \pi_j(y) p(y)$, hence

$$\begin{aligned}\delta^{B\pi}(y) &= \arg \min_{i \in \{0,\dots,M-1\}} \sum_{j=0}^{N-1} C_{ij} \pi_j(y) p(y) \\ &= \arg \min_{i \in \{0,\dots,M-1\}} \sum_{j=0}^{N-1} C_{ij} \pi_j(y)\end{aligned}$$

since $p(y)$ does not affect the minimizer. What does this mean? $\sum_{j=0}^{N-1} C_{ij} \pi_j(y)$ the average cost of choosing hypothesis $\mathcal{H}_i$ given $Y = y$, i.e. the **posterior** cost of choosing $\mathcal{H}_i$. The Bayes decision rule chooses the hypothesis that yields the minimum expected posterior cost.

Bayesian Hypothesis Testing with UCA: Part 1

The uniform cost assignment (UCA):

$$C_{ij} = \begin{cases} 0 & \text{if } x_j \in \mathcal{H}_i \\ 1 & \text{otherwise} \end{cases}$$

The conditional risk $R_j(\rho)$ under the UCA is simply the probability of not choosing the hypothesis that contains x_j, i.e. the probability of error when the state is x_j. The Bayes risk in this case is

$$r(\rho, \pi) = \sum_{j=0}^{N-1} \pi_j R_j(\rho) = \mathrm{Prob}(\text{error}).$$

Bayesian Hypothesis Testing with UCA: Part 2

Under the UCA, a Bayes decision rule can be written in terms of the posterior probabilities as

$$\begin{aligned}
\delta^{B\pi}(y) &= \arg \min_{i \in \{0,\dots,M-1\}} \sum_{j=0}^{N-1} C_{ij} \pi_j(y) \\
&= \arg \min_{i \in \{0,\dots,M-1\}} \sum_{x_j \notin \mathcal{H}_i} \pi_j(y) \\
&= \arg \min_{i \in \{0,\dots,M-1\}} \left[1 - \sum_{x_j \in \mathcal{H}_i} \pi_j(y) \right] \\
&= \arg \max_{i \in \{0,\dots,M-1\}} \sum_{x_j \in \mathcal{H}_i} \pi_j(y)
\end{aligned}$$

Hence, for hypothesis tests under the UCA, the Bayes decision rule is the MAP (maximum a posteriori) decision rule. When the hypothesis test is simple, $\delta^{B\pi}(y) = \arg\max_i \pi_i(y)$.

Bayesian Hypothesis Testing with UCA and Uniform Prior

Under the UCA and a uniform prior, i.e. $\pi_j = 1/N$ for all $j = 0, \dots, N-1$

$$\begin{aligned}\delta^{B\pi}(y) &= \arg \max_{i \in \{0,\dots,M-1\}} \sum_{x_j \in \mathcal{H}_i} \pi_j(y) \\ &= \arg \max_{i \in \{0,\dots,M-1\}} \sum_{x_j \in \mathcal{H}_i} \pi_j p_j(y) \\ &= \arg \max_{i \in \{0,\dots,M-1\}} \sum_{x_j \in \mathcal{H}_i} p_j(y)\end{aligned}$$

since $\pi_j = 1/N$ does not affect the minimizer.

- In this case, $\delta^{B\pi}(y)$ selects the **most likely** hypothesis, i.e. the hypothesis which best explains the observation y.
- This is called the **maximum likelihood** (ML) decision rule.
- Which is better, MAP or ML?

Simple Binary Bayesian Hypothesis Testing: Part 1

We have two states x_0 and x_1 and two hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$. For each $y \in \mathcal{Y}$, our problem is to compute

$$m_y = \arg \min_{i \in \{0,1\}} g_i(y, \pi)$$

where

$$\begin{aligned} g_0(y,\pi) &= \pi_0 C_{00} p_0(y) + \pi_1 C_{01} p_1(y) \\ g_1(y,\pi) &= \pi_0 C_{10} p_0(y) + \pi_1 C_{11} p_1(y) \end{aligned}$$

We only have two things to compare. We can simplify this comparison:

$$\begin{aligned} g_0(y,\pi) \geq g_1(y,\pi) &\Leftrightarrow \pi_0 C_{00} p_0(y) + \pi_1 C_{01} p_1(y) \geq \pi_0 C_{10} p_0(y) + \pi_1 C_{11} p_1(y) \\ &\Leftrightarrow p_1(y)\pi_1(C_{01} - C_{11}) \geq p_0(y)\pi_0(C_{10} - C_{00}) \\ &\Leftrightarrow \frac{p_1(y)}{p_0(y)} \geq \frac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})} \end{aligned}$$

where we have assumed that $C_{01} > C_{11}$ to get the final result. The expression $L(y) := \frac{p_1(y)}{p_0(y)}$ is known as the **likelihood ratio**.

Simple Binary Bayesian Hypothesis Testing: Part 2

Given $L(y) := \frac{p_1(y)}{p_0(y)}$ and

$$\tau \quad := \quad \frac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})},$$

our Bayes decision rule is then simply

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } L(y) > \tau \\ 0/1 & \text{if } L(y) = \tau \\ 0 & \text{if } L(y) < \tau. \end{cases}$$

Remark:

- For any $y \in \mathcal{Y}$ that result in $L(y) = \tau$, the Bayes risk is the same whether we decide $\mathcal{H}_0$ or $\mathcal{H}_1$. You can deal with this by always deciding $\mathcal{H}_0$ in this case, or always deciding $\mathcal{H}_1$, or flipping a coin, etc.

Simple Binary Bayesian Hypothesis Testing with UCA

Uniform cost assignment:

$$\begin{aligned} C_{00} = C_{11} = 0 \\ C_{01} = C_{10} = 1 \end{aligned}$$

In this case, the discriminant functions are simply

$$\begin{aligned} g_0(y,\pi) &= \pi_1 p_1(y) = \pi_1(y)p(y) \\ g_1(y,\pi) &= \pi_0 p_0(y) = \pi_0(y)p(y) \end{aligned}$$

and a Bayes decision rule can be written in terms of the posterior probabilities as

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } \pi_1(y) > \pi_0(y) \\ 0/1 & \text{if } \pi_1(y) = \pi_0(y) \\ 0 & \text{if } \pi_1(y) < \pi_0(y). \end{cases}$$

In this case, it should be clear that the Bayes decision rule is the MAP (maximum a posteriori) decision rule.

Example: Coherent Detection of BPSK

Suppose a transmitter sends one of two scalar signals and the signals arrive at a receiver corrupted by zero-mean additive white Gaussian noise (AWGN) with variance σ^2. We want to use Bayesian hypothesis testing to determine which signal was sent.

Signal model conditioned on state x_j:

$$Y = a_j + \eta$$

where a_j is the scalar signal and $\eta \sim \mathcal{N}(0, \sigma^2)$. Hence

$$p_j(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y - a_j)^2}{2\sigma^2} \right)$$

Hypotheses:

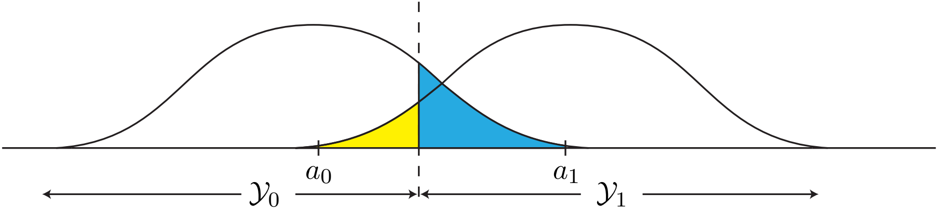
$$\begin{aligned} \mathcal{H}_0 &: a_0 \text{ was sent, or } Y \sim \mathcal{N}(a_0, \sigma^2) \\ \mathcal{H}_1 &: a_1 \text{ was sent, or } Y \sim \mathcal{N}(a_1, \sigma^2) \end{aligned}$$

Example: Coherent Detection of BPSK

This is just a simple binary hypothesis testing problem. Under the UCA, we can write

$$
\begin{aligned}
R_0(\rho) &= \int_{y \in \mathcal{Y}_1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y-a_0)^2}{2\sigma^2}\right) dy \\
R_1(\rho) &= \int_{y \in \mathcal{Y}_0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y-a_1)^2}{2\sigma^2}\right) dy
\end{aligned}
$$

where $\mathcal{Y}_j = \{y \in \mathcal{Y} : \rho_j(y) = 1\}$. Intuitively,



Example: Coherent Detection of BPSK

Given a prior π_0, $\pi_1 = 1 - \pi_0$, we can write the Bayes risk as

$$r(\rho, \pi) = \pi_0 R_0(\rho) + (1 - \pi_0) R_1(\rho)$$

The Bayes decision rule can be found by computing the likelihood ratio and comparing it to the threshold τ:

$$\begin{aligned} L(y) = \frac{p_1(y)}{p_0(y)} > \tau \quad & \Leftrightarrow \quad \frac{\exp\left(\frac{-(y-a_1)^2}{2\sigma^2}\right)}{\exp\left(\frac{-(y-a_0)^2}{2\sigma^2}\right)} > \frac{\pi_0}{\pi_1} \\ & \Leftrightarrow \quad \frac{(y-a_0)^2 - (y-a_1)^2}{2\sigma^2} > \ln \frac{\pi_0}{\pi_1} \\ & \Leftrightarrow \quad y > \frac{a_0 + a_1}{2} + \frac{\sigma^2}{a_1 - a_0} \ln \frac{\pi_0}{\pi_1} \end{aligned}$$

Example: Coherent Detection of BPSK

Let $\psi := \frac{a_0+a_1}{2} + \frac{\sigma^2}{a_1-a_0}\ln\frac{\pi_0}{\pi_1}$. Our Bayes decision rule is then :

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } y > \psi \\ 0/1 & \text{if } y = \psi \\ 0 & \text{if } y < \psi. \end{cases}$$

Using this decision rule, the Bayes risk is then

$$r(\delta^{B\pi}, \pi) = \pi_0 Q\left(\frac{\psi - a_0}{\sigma}\right) + (1-\pi_0) Q\left(\frac{a_1 - \psi}{\sigma}\right)$$

where $Q(x) := \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$. Remarks:

- When $\pi_0 = \pi_1 = \frac{1}{2}$, the decision boundary is simply $\frac{a_0+a_1}{2}$, i.e. the midpoint between a_0 and a_1 and $r(\delta^{B\pi}, \pi) = Q\left(\frac{a_1-a_0}{2\sigma}\right)$.
- When σ is very small, the prior has little effect on the decision boundary. When σ is very large, the prior becomes more important.

Composite Binary Bayesian Hypothesis Testing

We have $N > 2$ states $x_0, \dots, x_{N-1}$ and two hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ that form a valid partition on these states. For each $y \in \mathcal{Y}$, we compute

$$m_y = \arg\min_{i \in \{0,1\}} g_i(y,\pi), \text{ where } g_i(y,\pi) = \sum_{j=0}^{N-1} \pi_j C_{ij} p_j(y)$$

As was the case of simple binary Bayesian hypothesis testing, we only have two things to compare. The comparison boils down to

$$g_0(y,\pi) \geq g_1(y,\pi) \quad \Leftrightarrow \quad J(y) := \frac{\sum_j \pi_j C_{0j} p_j(y)}{\sum_j \pi_j C_{1j} p_j(y)} \geq 1.$$

Hence, a Bayes decision rule for the composite binary hypothesis testing problem is then simply

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } J(y) > 1 \\ 0/1 & \text{if } J(y) = 1 \\ 0 & \text{if } J(y) < 1. \end{cases}$$

Composite Binary Bayesian Hypothesis Testing with UCA

For $i = 0, 1$ and $j = 0, \dots, N-1$, the UCA is:

$$C_{ij} = \begin{cases} 0 & \text{if } x_j \in \mathcal{H}_i \\ 1 & \text{otherwise} \end{cases}$$

Hence, the discriminant functions are

$$g_i(y, \pi) = \sum_{x_j \in \mathcal{H}_i^c} \pi_j p_j(y) = 1 - \sum_{x_j \in \mathcal{H}_i} \pi_j p_j(y) = 1 - \sum_{x_j \in \mathcal{H}_i} \pi_j(y) p(y)$$

Under the UCA, the hypothesis test comparison boils down to

$$g_0(y, \pi) \geq g_1(y, \pi) \quad \Leftrightarrow \quad J(y) := \frac{\sum_{x_j \in \mathcal{H}_1} \pi_j(y)}{\sum_{x_j \in \mathcal{H}_0} \pi_j(y)} \geq 1.$$

and the Bayes decision rule $\delta^{B\pi}(y)$ is the same as before. Note that

$$J(y) := \frac{\sum_{x_j \in \mathcal{H}_1} \pi_j(y)}{\sum_{x_j \in \mathcal{H}_0} \pi_j(y)} = \frac{\mathrm{Prob}(\mathcal{H}_1 | y)}{\mathrm{Prob}(\mathcal{H}_0 | y)}$$

so, as before, the Bayes decision rule is the MAP decision rule.

Final Remarks on Bayesian Hypothesis Testing

1. A Bayes decision rule minimizes the Bayes risk $r(\rho, \pi) = \sum_j \pi_j R_j(\rho)$ under the prior state distribution π.
2. At least one deterministic decision rule $\delta^{B\pi}(y)$ achieves the minimum Bayes risk for any prior state distribution π.
3. The prior state distribution π can also be thought of as a parameter.
   - By varying π, we can generate a family of Bayes decision rules $\delta^{B\pi}$ parameterized by π.
   - This family of Bayes decision rules can achieve any operating point on the optimal tradeoff surface (or ROC curve).
4. In some applications, a prior state distribution is difficult to determine (or the risks cannot be weighted against each other). It may be more appropriate to use N-P in those cases.