ECE531 Spring 2011 Final Exam Solution

1. (a) $P_{fp} = \text{Prob}(\text{decide } \mathcal{H}_1 \mid x = x_0 = 1)$

$$= \text{Prob}(z < \tfrac{1}{2} \mid x=1) + \text{Prob}(z > 1 \mid x=1)$$

$$= \int_0^{1/2} 4z^3 \, dz + 0$$

$$= z^4 \Big|_0^{1/2} = \frac{1}{16}$$

(b) $\beta(x) = \text{Prob}(\text{decide } \mathcal{H}_1 \mid x)$

$$= \text{Prob}(z < \tfrac{1}{2} \mid x) + \text{Prob}(z > 1 \mid x)$$

Need to consider 3 cases: i) $0 < x < \frac{1}{2}$

ii) $\frac{1}{2} \le x < 1$

iii) $1 \le x$

in case (i), we have

$$\beta(x) = 1 + 0 = 1$$

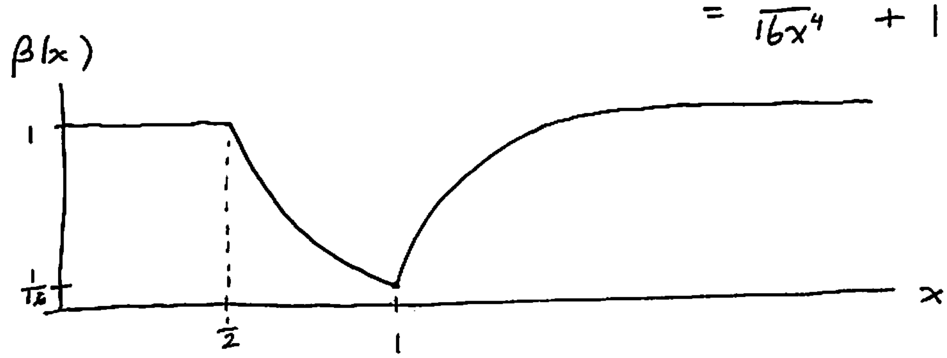
in case (ii), we have

$$\beta(x) = \int_0^{1/2} \frac{4z^3}{x^4} \, dz = \frac{1}{16x^4} \qquad \tfrac{1}{2} \le x < 1$$

in case (iii), we have

$$\beta(x) = \int_0^{1/2} \frac{4z^3}{x^4} \, dz + \int_1^x \frac{4z^3}{x^4} \, dz = \frac{1}{16x^4} + \frac{z^4}{x^4}\Big|_1^x$$

$$= \frac{1}{16x^4} + 1 - \frac{1}{x^4} = 1 - \frac{15}{16x^4}$$



2. a) Suppose you observed $y_0 = 0$, $y_1 = 1$, and $y_2 = 1$.
The only possible value that θ could be is $\theta = \frac{1}{2}$.
But the sample mean estimator will give $\hat{\theta} = \frac{2}{3}$, which contradicts the observation $y_0 = 0$ (this observation is impossible if $\theta = 2/3$).

b) The joint pdf of the observations, parameterized by θ, is

$$p(y;\theta) = \begin{cases} 1 & \theta - \frac{1}{2} \le y_k \le \theta + \frac{1}{2} \quad \text{for all } k = 0, \dots, n-1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \min(y) \ge \theta - \frac{1}{2} \quad \text{and} \quad \max(y) \le \theta + \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$p(y;\theta)$ is maximized for <u>any</u> θ satisfying

$$\boxed{\theta \le \min(y) + \tfrac{1}{2} \quad \text{and} \quad \theta \ge \max(y) - \tfrac{1}{2}}$$

Hence, an MLE could be

$$\boxed{\begin{aligned} \hat{\theta}_{MLE}(y) &= \frac{\min(y) + \frac{1}{2} + \max(y) - \frac{1}{2}}{2} \\ &= \frac{\min(y) + \max(y)}{2} \end{aligned}}$$

This is intuitively reasonable and avoids the problems of the sample mean estimator in part (a).

(c) This MLE is clearly not unique because the likelihood function is flat (a consequence of the uniform pdf). There are an infinite number of MLEs in this case.

3. Since everything in this problem is linear and Gaussian, we know that BLUE = MVU.

$$Y = \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix}}_{H} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

$Y = H\theta + W$ where $W \sim N(0, C)$ with $C = I$

We know the BLUE is

$$\hat{\theta}_{BLUE}(y) = (H^T C^{-1} H)^{-1} H^T C^{-1} y$$

$$= (H^T H)^{-1} H^T y \quad \text{since } C = I$$

$$H^T H = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

Hence

$$\hat{\theta}_{MVU}(y) = \hat{\theta}_{BLUE}(y) = \frac{1}{6} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \frac{1}{6} H^T y.$$

<u>Interpretation</u>: The channel estimates are obtained by "matched filtering" the observations with the known channel inputs (at different delays) and scaling by the energy of those input sequences.

4. a) $\hat{\theta}(Y) = Y_0$ and $Y_0 \sim N(\theta, 1)$
hence $E[Y_0] = \theta$ and $\hat{\theta}(Y)$ is unbiased.

b) $$\hat{\theta}_{MVU?}(y) = E\{\hat{\theta}(Y) \mid T(y) = T(y)\}$$
$$= E\{Y_0 \mid Y_0 = y_0, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}\}$$
$$= y_0$$

c) We know the MVUE for θ is the sample mean, i.e

$$\hat{\theta}_{MVU}(y) = \frac{1}{n} \sum_{k=0}^{n-1} y_k$$

This estimator is unbiased and has variance $\frac{1}{n}$.

Our estimator above, $\hat{\theta}_{MVU?}(y) = y_0$ is unbiased but has variance $1 > \frac{1}{n}$ for $n \geq 2$.

The RBLS theorem does not give the MVUE in this case because the sufficient statistic is <u>not complete</u>. Completeness is a necessary condition for the RBLS Theorem. Suppose $f(Y) = Y_0 - Y_1$.
Then clearly $E_\theta[f(Y)] = E_\theta[Y_0] - E_\theta[Y_1] = \theta - \theta = 0$ but $\text{Prob}_\theta(f(Y) = 0) = 0 \neq 1$. So the reason we didn't get the MVUE estimator here, even though we followed RBLS, is because we didn't find a <u>complete</u> sufficient statistic.

5. a) $\hat{X}[0|-1] = 0$, $\Sigma[0|-1] = 1$ } from prior on $X[0]$

$F = G = H$

$K[0] = \frac{1}{2}$

$\hat{X}[0|0] = 0 + \frac{1}{2}(1 - 0) = \frac{1}{2}$ $\quad Y[0] = 1$

$\Sigma[0|0] = 1 - \frac{1}{2} = \frac{1}{2}$

$\hat{X}[1|0] = \frac{1}{2}$

$\Sigma[1|0] = \frac{1}{2} + 1 = 3/2$

$K[1] = (3/2)\left(\frac{3}{2} + 1\right)^{-1} = \frac{3}{5}$

$\hat{X}[1|1] = \frac{1}{2} + 3/5\left(-2 - \frac{1}{2}\right)$ $\quad Y[1] = -2$

$\boxed{= -1}$

b) $\hat{X}[1|1] = E\left[X[1] \,\middle|\, Y = y = \begin{bmatrix}1\\-2\end{bmatrix}\right]$

$= E[X[1]] + \mathrm{cov}(X[1], Y)\,\mathrm{cov}(Y, Y)^{-1}(y - E[Y])$

$X[1] = X[0] + u[0] \rightarrow E[X[1]] = 0$

$Y[0] = X[0] + v[0] \rightarrow E[Y[0]] = 0$

$Y[1] = X[1] + V[1] = X[0] + u[0] + V[1] \rightarrow E[Y[1]] = 0$

$\mathrm{cov}(X[1], Y) = \mathrm{cov}\left(X[0] + u[0], \begin{bmatrix} X[0] + v[0] \\ X[0] + u[0] + v[1] \end{bmatrix}\right)$

$= [1, 2]$

$\mathrm{cov}(Y, Y) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad \mathrm{cov}(Y, Y)^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

$\Rightarrow \hat{X}[1|1] = 0 + \frac{1}{5}[1 \quad 3]\begin{bmatrix}1\\-2\end{bmatrix} = \boxed{-1}$ agrees