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10.2.2 Exponential Families

It is often difficult to use the definitions to find minimal sufficient or complete sufficient statistics. However, most statistical models are examples of exponential families for which fairly simple criteria exist. We say that a random vector **X** with density function $f(\mathbf{x};\boldsymbol{\theta})$ has a *p-dimensional exponential family* if

$$f(\mathbf{x};\boldsymbol{\theta}) = h(\mathbf{x})k(\boldsymbol{\theta})\exp[d_1(\boldsymbol{\theta})T_1(\mathbf{x}) + \cdots + d_p(\boldsymbol{\theta})T_p(\mathbf{x})], \qquad \mathbf{x}\in A,$$

or equivalently, if

$$L_X(\boldsymbol{\theta}) = h(\mathbf{X})k(\boldsymbol{\theta})\exp[d_1(\boldsymbol{\theta})T_1(\mathbf{X}) + \cdots + d_p(\boldsymbol{\theta})T_p(\mathbf{X})], \qquad \mathbf{X}\in A, \tag{10–3}$$

where $h(\mathbf{X})$ and $k(\boldsymbol{\theta})$ are one-dimensional functions and

$$T(\mathbf{X}) = (T_1(\mathbf{X}),\ldots,T_p(\mathbf{X})) \quad\text{and}\quad d(\boldsymbol{\theta}) = (d_1(\boldsymbol{\theta}),\ldots,d_p(\boldsymbol{\theta}))$$

are p-dimensional functions. Note that the dimension of the family (or of $d(\boldsymbol{\theta})$) need not be the same as the dimension of $\boldsymbol{\theta}$. (See Example 10–3.) It is quite important that the set A of possible values of **X** does not depend on $\boldsymbol{\theta}$.

By the factorization criterion, $\mathbf{T} = T(\mathbf{X})$ is a sufficient statistic for this exponential family. In this section, we give conditions for **T** to be a minimal sufficient or complete sufficient statistic. We assume throughout that the components of $d(\boldsymbol{\theta})$ take values on an interval. For simplicity, we look first at one-dimensional exponential families, so that $d(\boldsymbol{\theta})$ and $T(\mathbf{X})$ are one dimensional.

Theorem 10–8. (Exponential criterion I) Under fairly general conditions, if **X** has a one-dimensional exponential family given in Equation (10–3), and if $d(\boldsymbol{\theta})$ takes values on an interval, then $T = T(\mathbf{X})$ is a complete sufficient statistic (and is therefore also minimal sufficient).

Proof. A proof of the theorem in its full generality is beyond the level of this book, but can be found in Lehmann (1986), pp. 142–143. We give a proof for the case in which T is a nonnegative integer-valued random variable. In that case, let A_t be the set of all **x**'s such that $T(\mathbf{x}) = t$. Then

$$P(T=t) = P(\mathbf{X}\in A_t) = \sum_{A_t} h(\mathbf{x})k(\boldsymbol{\theta})\exp[T(\mathbf{x})d(\boldsymbol{\theta})]$$

$$= k(\boldsymbol{\theta})\exp(td(\boldsymbol{\theta}))\sum_{A_t} h(\mathbf{x}) = h^*(t)k(\boldsymbol{\theta})\exp[td(\boldsymbol{\theta})].$$

(Note that $T(\mathbf{x}) = t$ for all $\mathbf{x}\in A_t$.) Therefore, if $E_{\boldsymbol{\theta}}g(T) = 0$, then

$$0 = \sum_{t=0}^{\infty} g(t)h^*(t)k(\boldsymbol{\theta})(\exp(d(\boldsymbol{\theta})))^t = k(\boldsymbol{\theta})\sum_{t=0}^{\infty} g(t)h^*(t)\exp(d(\boldsymbol{\theta}))^t.$$

Now, let $\delta = \exp(d(\boldsymbol{\theta}))$. Then

$$\sum_{t=0}^{\infty} g(t)h^*(t)\delta^t = 0 = \sum_{t=0}^{\infty} 0\delta^t.$$

Since δ takes on values in an interval, it follows by the uniqueness of power series that

$$g(t)h^*(t) = 0$$

for all integers t, and hence,

$$P(g(T)=0) = 1.$$

Therefore, T is a complete sufficient statistic. □

Example A1

Suppose we observe $X_i \sim P(i\theta)$, independent. Then

$$L_X(\theta) = h_1(\mathbf{X})\exp[-\theta a_n]\exp[(\log\theta)\textstyle\sum X_i],$$

which is in the form of Theorem 10–8, with $T(\mathbf{X}) = \Sigma X_i$. Note that the space of possible values for θ is the set of all positive numbers, so that the space of possible values for $\delta = \log(\theta)$ is all numbers, which contains an interval. Therefore, $T = T(\mathbf{X})$ is a complete sufficient statistic for this model. Also, the MLE P is an invertible function of T and is a complete sufficient statistic for the model.

Example A2

Suppose we observe $X_i \sim N(i\theta, 1)$, independent. Then

$$L_X(\theta) = h_2(\mathbf{X})\exp\left(\frac{-\theta^2 b_n}{2}\right)\exp(\theta(\textstyle\sum iX_i)),$$

which is in exponential form with $T(\mathbf{X}) = \Sigma iX_i$. Therefore, $T = T(\mathbf{X})$ is a complete sufficient statistic for this model. Also, the MLE Q is an invertible function of T and is thus complete and sufficient.

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