ECE531 Spring 2011 Final Examination

Instructions: This exam is worth a total of 500 points. You may consult two double-sided letter-sized sheets of notes (in your own handwriting) and you may use a calculator during the exam. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution. The exam is closed-book.

1. 100 points. Suppose Y_0, Y_1, Y_2, Y_3 are i.i.d. observations drawn from $\mathcal{U}(0, x)$ where $x > 0$ is an unknown deterministic constant. You have two hypotheses:

$$\begin{aligned}\mathcal{H}_0 &: x = 1\\ \mathcal{H}_1 &: x \neq 1.\end{aligned}$$

   Someone proposes to use the following detector:

$$\rho(Y_0, Y_1, Y_2, Y_3) = \begin{cases} 1 & \max Y_i < \frac{1}{2} \text{ or } \max Y_i > 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) (40 points) Compute the false positive probability of this decision rule.

   (b) (60 points) Compute and sketch the power function $\beta(x)$ of this decision rule.

   Hint: Let $Z = \max Y_i$. The pdf of Z parameterized by x is

$$p_x(z) = p_Z(z\,;\,x) = \begin{cases} \frac{4z^3}{x^4} & 0 \leq z \leq x \\ 0 & \text{otherwise.} \end{cases}$$

2. 100 points. Suppose we observe the sequence $Y_0, Y_1, \ldots, Y_{n-1}$ given by

$$Y_k = \theta + U_k$$

   for $k = 0, \ldots, n-1$ where $U_k \overset{\text{i.i.d.}}{\sim} \mathcal{U}(-0.5, 0.5)$ and $\theta \in \mathbb{R}$ is an unknown non-random parameter.

   (a) (25 points) Explain why the sample mean $\hat{\theta}(y) = \frac{1}{n}\sum_{k=0}^{n-1} y_k$ may not be a "good" estimator of θ in this problem. Hint: Consider an example with three observations.

   (b) (50 points) Find a/the maximum-likelihood estimator (MLE) of θ.

   (c) (25 points) Is your MLE unique? Explain.

3. 100 points. Consider the scenario shown in Figure 1 with the following notation:

   - The known input sequence is $s_0, s_1, \ldots$.
   - The unknown non-random channel impulse response is $\theta = [\theta_1, \theta_2]^\top \in \mathbb{R}^2$.

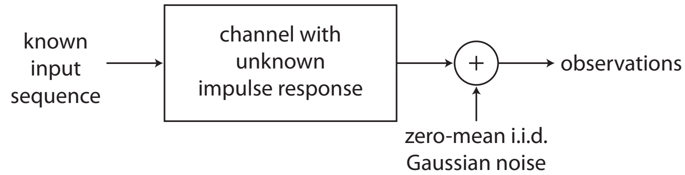

Figure 1: Channel estimation problem scenario.

- The Gaussian noise is $W_0, W_1, \dots$ with $W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.
- The observations are $Y_0, Y_1, \dots$.

Suppose $[s_0, s_1, s_2] = [1, 2, -1]$, i.e. a known input sequence of three numbers was sent through the channel. Note that there will be four observations Y_0, Y_1, Y_2, Y_3 in this case because the output of the channel is simply the convolution of the input sequence with the unknown non-random two-sample channel impulse response. Find the MVUE of the unknown channel impulse response θ.

4. 100 points. Suppose you observe samples $Y_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, 1)$ for $k = 0, 1, \dots, n-1$ where $\theta \in \mathbb{R}$ is a scalar unknown random parameter. A sufficient statistic for θ is, of course, $T(Y) = [Y_0, \dots, Y_{n-1}]^\top$ since the whole observation is always sufficient. Suppose someone suggests to use $\hat{\theta}(Y) = Y_0$ as "any old unbiased estimator" and to apply the Rao-Blackwell-Lehmann-Sheffe theorem to compute the MVUE by evaluating the conditional expectation

$$\hat{\theta}_{\mathsf{MVU?}}(y) = \mathrm{E}\left\{\hat{\theta}(Y) \,|\, T(Y) = T(y)\right\}. \tag{1}$$

   (a) (25 points) Confirm $\hat{\theta}(Y) = Y_0$ is an unbiased estimator of θ.
   (b) (25 points) Compute $\hat{\theta}_{\mathsf{MVU?}}(y)$ in (1).
   (c) (50 points) Explain why $\hat{\theta}_{\mathsf{MVU?}}(y)$ in (1) is not the MVUE of θ. Why doesn't the RBLS theorem give the MVUE in this case?

5. 100 points. Consider the *scalar* dynamical system

$$X[\ell+1] \quad = \quad X[\ell] + U[\ell] \text{ for } \ell = 0, 1, \dots$$

with observations given by $Y[\ell] = X[\ell] + V[\ell]$ for $\ell = 0, 1, \dots$. Assume that $\{U[0], U[1], \dots\}$ and $\{V[0], V[1], \dots\}$ are independent sequences of i.i.d. Gaussian random variables with $U[k] \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and $V[k] \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. Also assume that the initial state $X[0] \sim \mathcal{N}(0,1)$ and is independent of all $U[\ell]$ and $V[\ell]$.

   (a) (50 points) Given $Y[0] = 1$ and $Y[1] = -2$ compute the Kalman filter state estimate $\hat{X}[1\,|\,1]$, i.e. the Kalman filter estimate of the state $X[1]$ given the observations $Y = [Y[0], Y[1]]^\top$.
   (b) (50 points) Confirm that $\hat{X}[1\,|\,1]$ is the MMSE estimate of the state $X[1]$ given the observations $Y = [Y[0], Y[1]]^\top$ by directly computing the MMSE estimate via the conditional expectation, i.e. compute

$$\hat{X}_{\mathsf{mmse}} = \mathrm{E}\{X[1]\,|\,Y[0], Y[1]\}.$$