ECE531 Homework Assignment Number 10 Solution

Due by 8:50pm on Thursday 20-Apr-2011

Make sure your reasoning and work are clear to receive full credit for each problem.

1. 5 points. Prove that the innovation

$$\tilde{Y}[\ell+1\,|\,\ell] := Y[\ell+1] - H[\ell+1]\hat{X}[\ell+1\,|\,\ell]$$

is a zero-mean Gaussian random vector uncorrelated with $Y[0], \ldots, Y[\ell]$. Also prove that the innovation sequence is white.

Solution: To prove that the innovation sequence is a sequence of Gaussian random vectors, first note that $\hat{X}[\ell+1\,|\,\ell]$ is a linear transformation on $\mathcal{Y}_0^\ell = [Y^\top[0], \ldots, Y^\top[\ell]]^\top$ and that $\mathcal{Y}_0^\ell$ is jointly Gaussian. Since

$$\tilde{Y}[\ell+1\,|\,\ell] = \underbrace{\begin{bmatrix} Y & H[\ell+1] \end{bmatrix}}_{\text{linear transformation}} \begin{bmatrix} Y[\ell+1] \\ \hat{X}[\ell+1\,|\,\ell] \end{bmatrix} = A\mathcal{Y}_0^{\ell+1}$$

is also a linear transformation on a Guassian random vector, the innovation sequence is a sequence of Gaussian random vectors.

So show that the innovations are zero mean, we will first show that

$$\mathrm{E}\{Y[\ell+1]\,|\mathcal{Y}_0^\ell\} = H[\ell+1]\hat{X}[\ell+1\,|\,\ell].$$

To see this, we can write

$$\begin{aligned}
\mathrm{E}\{Y[\ell+1]\,|\mathcal{Y}_0^\ell\} &= \mathrm{E}\left\{H[\ell+1]X[\ell+1] + V[\ell+1]\,|\,\mathcal{Y}_0^\ell\right\} \\
&= H[\ell+1]\mathrm{E}\left\{X[\ell+1]\,|\,\mathcal{Y}_0^\ell\right\} + \mathrm{E}\left\{V[\ell+1]\,|\,\mathcal{Y}_0^\ell\right\} \\
&= H[\ell+1]\hat{X}[\ell+1\,|\,\ell] + 0
\end{aligned}$$

since $\mathcal{Y}_0^\ell$ is irrelevant to $V[\ell+1]$ and V is zero mean. This result establishes that the innovation can be written as

$$\tilde{Y}[\ell+1\,|\,\ell] = Y[\ell+1] - \mathrm{E}\{Y[\ell+1]\,|\mathcal{Y}_0^\ell\}.$$

Now we can compute the expectation

$$\begin{aligned}
\mathrm{E}\left(\tilde{Y}[\ell+1\,|\,\ell]\right) &= \mathrm{E}\Big(Y[\ell+1] - \mathrm{E}\left\{Y[\ell+1]\,|\mathcal{Y}_0^\ell\right\}\Big) \\
&= \mathrm{E}(Y[\ell+1]) - \mathrm{E}(Y[\ell+1]) \\
&= 0
\end{aligned}$$

where we used the iterated expectation property of conditional expectations, i.e. $\mathrm{E}(\mathrm{E}(Y|X)) = \mathrm{E}(Y)$.

Now to show the sequence is white, suppose $n < \ell$. We can use the fact that the innovations are zero mean to write

$$\begin{aligned}
\mathrm{cov}\left\{\tilde{Y}[\ell+1\,|\,\ell], \tilde{Y}[n+1\,|\,n]\right\} &= \mathrm{E}\left\{\tilde{Y}[\ell+1\,|\,\ell]\tilde{Y}^\top[n+1\,|\,n]\right\} \\
&= \mathrm{E}\left\{\mathrm{E}\left\{\tilde{Y}[\ell+1\,|\,\ell]\tilde{Y}^\top[n+1\,|\,n]\,|\,\mathcal{Y}_0^n\right\}\right\}
\end{aligned}$$

where we've again used the iterated expectation property of conditional expectations. Conditioning on $\mathcal{Y}_0^n$ in the inner expectation causes $\tilde{Y}^\top[n+1\,|\,n]$ to become a non-random constant, so we pop it out of the inner expectation to write

$$\mathrm{E}\left\{\mathrm{E}\left\{\tilde{Y}[\ell+1\,|\,\ell]\tilde{Y}^\top[n+1\,|\,n]\,|\,\mathcal{Y}_0^n\right\}\right\} = \mathrm{E}\left\{\mathrm{E}\left\{\tilde{Y}[\ell+1\,|\,\ell]\,|\,\mathcal{Y}_0^n\right\}\tilde{Y}^\top[n+1\,|\,n]\right\}.$$

Let's look at that inner conditional expectation first.

$$\begin{aligned}
\mathrm{E}\left\{\tilde{Y}[\ell+1\,|\,\ell]\,|\,\mathcal{Y}_0^n\right\} &= \mathrm{E}\left\{Y[\ell+1] - \mathrm{E}\{Y[\ell+1]\,|Y[\ell]\}\,|\,\mathcal{Y}_0^n\right\} \\
&= \mathrm{E}\Big\{Y[\ell+1]\,|\,\mathcal{Y}_0^n\Big\} - \mathrm{E}\Big\{\mathrm{E}\{Y[\ell+1]\,|Y[\ell]\}\,|\,\mathcal{Y}_0^n\}\Big\} \\
&= \mathrm{E}\Big\{Y[\ell+1]\,|\,\mathcal{Y}_0^n\Big\} - \mathrm{E}\Big\{\{Y[\ell+1]\,|\,\mathcal{Y}_0^n\}\Big\} \\
&= 0.
\end{aligned}$$

The same approach can be used to show the same result for $n > \ell$. So $\mathrm{cov}\left\{\tilde{Y}[\ell+1\,|\,\ell], \tilde{Y}[n+1\,|\,n]\right\} = 0$ unless $n = \ell$, which proves the innovations sequence is white.

Finally, we just need to show that the innovation $\tilde{Y}[\ell+1\,|\,\ell]$ is uncorrelated with $\mathcal{Y}_0^\ell = [Y^\top[0], \ldots, Y^\top[\ell]]^\top$. For notational convenience, let's let

$$X = Y[\ell+1]$$

and

$$Z = \mathcal{Y}_0^\ell.$$

The innovation is then

$$\tilde{Y}[\ell+1\,|\,\ell] = X - \mathrm{E}\{X\,|Z\}.$$

To show that the innovation is uncorrelated with the past observations, we need to show the correlation matrix is zero, i.e.

$$\mathrm{E}\left\{(X - \mathrm{E}\{X\,|Z\})\,Z^\top\right\} = 0.$$

To see this, we let $\mu_X = \mathrm{E}[X]$, $\mu_Z = \mathrm{E}[Z]$, $\Sigma_{X,Z} := \mathrm{cov}(X,Z)$, $\Sigma_{Z,Z} := \mathrm{cov}(Z,Z)$, and use the conditional mean result for jointly Gaussian random vectors to write

$$\begin{aligned}
\mathrm{E}\left\{(X - \mathrm{E}\{X\,|Z\})\,Z^\top\right\} &= \mathrm{E}\left\{\Big(X - \mathrm{E}[X] - \Sigma_{X,Z}\Sigma_{Z,Z}^{-1}(Z - \mathrm{E}[Z])\Big)\,Z^\top\right\} \\
&= \mathrm{E}\left\{\Big(X - \mu_X - \Sigma_{X,Z}\Sigma_{Z,Z}^{-1}(Z - \mu_Z)\Big)\,Z^\top\right\} \\
&= \mathrm{E}\left\{XZ^\top\right\} - \mu_X\mathrm{E}\left\{Z^\top\right\} - \Sigma_{X,Z}\Sigma_{Z,Z}^{-1}\mathrm{E}\left\{ZZ^\top\right\} + \Sigma_{X,Z}\Sigma_{Z,Z}^{-1}\mu_Z\mathrm{E}\left\{Z^\top\right\} \\
&= \Sigma_{X,Z} + \mu_X\mu_Z^\top - \mu_X\mu_Z^\top - \Sigma_{X,Z}\Sigma_{Z,Z}^{-1}\left(\Sigma_{Z,Z} + \mu_Z\mu_Z^\top\right) + \Sigma_{X,Z}\Sigma_{Z,Z}^{-1}\mu_Z\mu_Z^\top \\
&= 0.
\end{aligned}$$

Hence the innovation $\tilde{Y}[\ell+1\,|\,\ell]$ is uncorrelated with $\mathcal{Y}_0^\ell = [Y^\top[0], \ldots, Y^\top[\ell]]^\top$.

2. 5 points. Suppose you have a scalar-state, scalar-observation linear time-invariant dynamic model given by

$$x[n+1] = fx[n] + gu[n]$$
$$y[n] = hx[n] + v[n]$$

for $n = 0, 1, \ldots$ with f, g, h all non-zero scalars. The process noise $u[n]$ is a zero-mean Gaussian random process with $\mathrm{E}[u[m]u[n]] = q\delta_{m,n}$ with $q > 0$. The measurement noise $v[n]$ is a zero-mean Gaussian random process with $\mathrm{E}[v[m]v[n]] = r\delta_{m,n}$ with $r > 0$ and is assumed to be uncorrelated with $\{u[0], u[1], \ldots\}$. Write the Kalman filter recursion and determine $\lim_{\ell\to\infty} \Sigma[\ell+1 \,|\, \ell]$ and $\lim_{\ell\to\infty} \Sigma[\ell \,|\, \ell]$, i.e. the steady-state prediction and estimation covariances. Interpret your results.

Solution: In the limit, if it exists, the prediction ECMs and estimation ECMs converge to constants such that $\Sigma[\ell+1 \,|\, \ell] = \Sigma[\ell \,|\, \ell-1] \equiv p$ and $\Sigma[\ell \,|\, \ell] = \Sigma[\ell-1 \,|\, \ell-1] \equiv s$. Using this notation, we can write the KF recursion for the ECMs in steady state as

$$p = f^2 s + g^2 q$$
$$s = p - hK[\ell]p$$

with

$$\begin{aligned} K[\ell] &= \Sigma[\ell \,|\, \ell-1]H^\top \left(H\Sigma[\ell \,|\, \ell-1]H^\top + R\right)^{-1} \\ &= \frac{h\Sigma[\ell \,|\, \ell-1]}{h^2\Sigma[\ell \,|\, \ell-1] + r} \\ &= \frac{hp}{h^2 p + r} \end{aligned}$$

Plugging this back into the KF recursion for the error covariance matrices, we have

$$\begin{aligned} p &= f^2 s + q' \\ s &= p - \frac{h^2 p}{h^2 p + r} = \frac{rp}{h^2 p + r} = \frac{r' p}{p + r'} \end{aligned}$$

where $q' := g^2 q$ and $r' := r/h^2$. These two equations can be combined to write

$$p = f^2 \frac{r' p}{p + r'} + q'.$$

This can be rearranged to write

$$p^2 + (r' - f^2 r' - q')p - q' r' = 0.$$

We can use the quadratic equation to solve for p. The solutions for p are then

$$p = \frac{(-r' + f^2 r' + q') \pm \sqrt{(-r' + f^2 r' + q')^2 + 4q' r'}}{2}.$$

Note that the discriminant is positive, so there will be two distinct real roots. The square root of the discriminant is also larger than $|r' - f^2 r' - q'|$, so the unique positive root is then

$$p = \frac{(-r' + f^2 r' + q') + \sqrt{(-r' + f^2 r' + q')^2 + 4q' r'}}{2}$$

As a sanity check, let's let the measurement noise variance go to zero, i.e. $r' \to 0$. In this case, we have

$$\lim_{r'\to 0} p = \frac{q' + q'}{2} = q'.$$

This makes sense because the variance of the prediction error is equal to the process noise variance.

We can use the same process for the estimator error variance to write

$$s = \frac{r'(f^2 s + q')}{(f^2 s + q') + r'}.$$

This can be rearranged to write

$$f^2 s^2 + (r' - f^2 r' + q')s - q'r' = 0.$$

We can use the quadratic equation to solve for s. The unique positive solution for s is then

$$s = \frac{(-r' + f^2 r' - q') + \sqrt{(-r' + f^2 r' - q')^2 + 4f^2 q' r'}}{2f^2}.$$

If we perform the same sanity check,

$$\lim_{r' \to 0} s = \frac{-q' + q'}{2f^2} = 0.$$

This makes sense since, in the absence of measurement noise, the state estimation error should be zero.

As a final check, you can confirm that $p = f^2 s + q'$. It turns out that the discriminant for p is the same as the discriminant for s, and once you show this then it is easy to see that $p = f^2 s + q'$.

For these limits to exist, we need $h \neq 0$, otherwise the observations are just measurement noise and don't tell us anything about the state. When the observations tell us nothing about the state, the prediction and estimation errors can blow up. The Kalman gain also goes to zero, which says that the observations are useless.

3. 5 points. Kay I: Problem 13.10

Solution: In this problem, the state $X[n] = A$ is *not* dynamic. As discussed in lecture, this means we don't need to generate predictions anymore and the KF recursion can be simplified to

$$\begin{aligned} K[\ell] &= \Sigma[\ell-1]H^\top \left(H\Sigma[\ell-1]H^\top + R\right)^{-1} \\ \hat{X}[\ell] &= \hat{X}[\ell-1] + K[\ell]\left(Y[\ell] - H\hat{X}[\ell-1]\right) \\ \Sigma[\ell] &= \Sigma[\ell-1] - K[\ell]H\Sigma[\ell-1] \end{aligned}$$

with $\hat{X}[-1] = 0$, $\Sigma[-1] = \sigma_A^2$, $H = 1$, and $R = \sigma^2$. Since everything is scalar, the Kalman gain is $K[\ell] = \frac{\Sigma[\ell-1]}{\Sigma[\ell-1]+\sigma^2}$ and the recursion can be simplified to

$$\begin{aligned} \hat{X}[\ell] &= \hat{X}[\ell-1] + \frac{\Sigma[\ell-1]}{\Sigma[\ell-1]+\sigma^2}\left(Y[\ell] - \hat{X}[\ell-1]\right) \\ \Sigma[\ell] &= \Sigma[\ell-1] - \frac{\Sigma^2[\ell-1]}{\Sigma[\ell-1]+\sigma^2} = \frac{\Sigma[\ell-1]\sigma^2}{\Sigma[\ell-1]+\sigma^2}. \end{aligned}$$

Let's look at the evolution of $\Sigma[\ell]$ (the minimum MSE) first. Starting with $\Sigma[-1] = \sigma_A^2$, we see that

$$\begin{aligned} \Sigma[0] &= \frac{\sigma_A^2\sigma^2}{\sigma_A^2+\sigma^2} \\ \Sigma[1] &= \frac{\sigma_A^2\sigma^2}{2\sigma_A^2+\sigma^2} \\ \Sigma[2] &= \frac{\sigma_A^2\sigma^2}{3\sigma_A^2+\sigma^2} \\ &\vdots \\ \Sigma[\ell-1] &= \frac{\sigma_A^2\sigma^2}{\ell\sigma_A^2+\sigma^2} \quad \text{for all } \ell = 1, 2, \dots. \end{aligned}$$

So now we have an explicit expression for the variance of the estimate at each time step and we see that it is decreasing in ℓ as we should expect.

Now let's look at the evolution of $X[\ell]$. We can compute the Kalman gain explicitly as

$$K[\ell] = \frac{\Sigma[\ell-1]}{\Sigma[\ell-1]+\sigma^2} = \frac{\sigma_A^2}{(\ell+1)\sigma_A^2+\sigma^2}$$

hence

$$\begin{aligned} \hat{X}[\ell] &= \hat{X}[\ell-1] + \frac{\sigma_A^2}{(\ell+1)\sigma_A^2+\sigma^2}\left(Y[\ell] - \hat{X}[\ell-1]\right) \\ &= \left(1 - \frac{\sigma_A^2}{(\ell+1)\sigma_A^2+\sigma^2}\right)\hat{X}[\ell-1] + \frac{\sigma_A^2}{(\ell+1)\sigma_A^2+\sigma^2}Y[\ell] \\ &= a[\ell-1]\hat{X}[\ell-1] + b[\ell-1]Z[\ell-1] \end{aligned}$$

where $Z[\ell-1] := Y[\ell]$. Written in this form, you should recognize that this is a discrete-time, linear, time-varying state update equation. Given an initial condition $\hat{X}[-1] = 0$, it has a unique solution. As shown in ECE504 (and adapting the notation to this problem), the solution to this LTV system is

$$\hat{X}[\ell] = \Phi[\ell,-1]\hat{X}[-1] + \sum_{k=-1}^{\ell-1}\Phi[\ell,k+1]b[k]Z[k]$$

where

$$\Phi[m,n] := \begin{cases} \text{undefined} & m < n \\ 1 & m = n \\ a[m-1]a[m-2]\cdots a[n] & m > n \end{cases}$$

is the discrete-time state transition matrix. Since $\hat{X}[-1] = 0$, we just need to work on

$$\begin{aligned}\hat{X}[\ell] &= \sum_{k=-1}^{\ell-1} \Phi[\ell, k+1] b[k] Y[k+1] \\ &= (a[\ell-1]\cdots a[0])b[-1]Y[0] + (a[\ell-1]\cdots a[1])b[0]Y[1] + (a[\ell-1]\cdots a[2])b[1]Y[2] + \cdots \\ &\quad + a[\ell-1]b[\ell-2]Y[\ell-1] + b[\ell-1]Y[\ell]\end{aligned}$$

Note that

$$a[\ell-1] = \frac{\ell\sigma_A^2 + \sigma^2}{(\ell+1)\sigma_A^2 + \sigma^2}$$

and

$$b[\ell-2] = \frac{\sigma_A^2}{\ell\sigma_A^2 + \sigma^2}$$

hence

$$a[\ell-1]b[\ell-2] = \frac{\sigma_A^2}{(\ell+1)\sigma_A^2 + \sigma^2} = b[\ell-1].$$

Hence, each product

$$a[\ell-1]\cdots a[m]b[m-1] = b[\ell-1]$$

and the estimator can be simplified to

$$\begin{aligned}\hat{X}[\ell] &= b[\ell-1]\sum_{k=0}^{\ell} Y[k] \\ &= \frac{\sigma_A^2}{(\ell+1)\sigma_A^2 + \sigma^2}\sum_{k=0}^{\ell} Y[k] \\ &= \frac{\sigma_A^2}{\sigma_A^2 + \frac{\sigma^2}{\ell+1}}\bar{Y}\end{aligned}$$

where $\bar{Y}$ is the sample mean (with $\ell + 1$ samples). This can be confirmed to agree with the conditional mean (the Bayesian MMSE estimate).

4. 10 points. Kay I: Problem 13.11. To clarify what I am expecting here, I would like you to write some Matlab/Octave code to run the KF over lots of realizations of the process noise and measurement noise, generate Monte-Carlo results for the MSE, and compare the Monte-Carlo MSE results to the error covariance matrices (actually scalars in this problem). You can also use your results from Problem 2 to confirm your KF is working correctly in part (b).

   Solution: Here is my code (you change line 40 for parts a, b, and c).

```
1  % ECE531 HW10 problem 4
2  % DRB 13-Apr-2011
3  % ----------------------------------------------------------------------
4  % USER PARAMETERS BELOW
5  % ----------------------------------------------------------------------
6  N = 51;                                % number of states to generate (n=0,dots,N-1)
7  mu_s = 0;                              % mean of initial state (Kay notation)
8  sig2_s = 1;                            % variance of initial state (Kay notation)
9  a = 0.9;                               % state update parameter (Kay notation)
10 sig2_u = 1;                            % variance of process noise
11 iterations = 5000;                     % number of iterations for Monte-Carlo averaging
12 % ----------------------------------------------------------------------
13 
14 % Kay's scalar-state, scalar-observation dynamical model
15 F = a;                                 % state update matrix
16 G = 1;                                 % input matrix
17 H = 1;                                 % output matrix
18 Q = sig2_u;                            % process noise variance
19 X = zeros(1,N);                        % states
20 Xp = zeros(1,N);                       % predictions
21 Xe = zeros(1,N);                       % estimates
22 Sp = zeros(1,N);                       % prediction ECM
23 Se = zeros(1,N);                       % estimation ECM
24 Ee = zeros(iterations,N);              % actual squared estimation error
25 Ep = zeros(iterations,N);              % actual squared prediction error
26 K = zeros(1,N);                        % kalman gain
27 
28 % initialize the KF
29 Xp(1) = mu_s;                          % this is Xhat[0 | -1]
30 Sp(1) = sig2_s;                        % this is Sigma[0 | -1]
31 
32 for i=1:iterations,
33 
34     % generate initial state
35     X(1) = sqrt(sig2_s)*randn+mu_s;        % initial state
36     Ep(i,1) = (Xp(1)-X(1))^2;              % squared prediction error
37 
38     for n=1:N,         % sample index
39 
40         R = (1.1)^(n+1);                       % variance of measurement noise
41         V = sqrt(R)*randn;                     % measurement noise realization
42         Y(n) = H*X(n)+V;
43 
44         % KALMAN FILTER (5 steps as shown in lecture)
45         K(n) = Sp(n)*H'*inv(H*Sp(n)*H'+R);  % Kalman gain
46         Xe(n) = Xp(n)+K(n)*(Y(n)-H*Xp(n));  % estimate
47         Ee(i,n) = (Xe(n)-X(n))^2;           % squared estimation error
48         Se(n) = Sp(n) - K(n)*H*Sp(n);       % ECM of estimate
49 
50         if n<N,
51             U = sqrt(sig2_u)*randn;         % process noise
52             X(n+1) = F*X(n)+G*U;            % update state (dynamical model)
53             Xp(n+1) = F*Xe(n);              % prediction
54             Ep(i,n+1) = (Xp(n+1)-X(n+1))^2; % squared prediction error
55             Sp(n+1) = F*Se(n)*F'+Q;         % ECM of prediction
56         end
57 
58     end
59 
60 end
61 
62 figure(1)
```

```
plot(0:N-1,Xp,'-+',0:N-1,Xe,'-s',0:N-1,X,'-o');
xlabel('sample␣index');
ylabel('signal␣level');
legend('predicted␣state','estimated␣state','true␣state');

figure(2)
plot(0:N-1,mean(Ep),'-+',0:N-1,mean(Ee),'-s');
hold on
plot(0:N-1,Sp,'--',0:N-1,Se,'--');
hold off
xlabel('sample␣index');
ylabel('MSE');
legend('predicted␣state␣(monte-carlo)','estimated␣state␣(monte-carlo)',...
    'prediction␣ECM','estimation␣ECM' );

figure(3)
plot(0:N-1,K);
xlabel('sample␣index');
ylabel('Kalman␣Gain');
```

In part (a), we have an example output shown in Figure 1, the Kalman gain shown in Figure 2, and the MSE of the predictions and estimates shown in Figure 3. We see that the Kalman gain goes to one because the observations become less noisy and the estimates get better as time progresses. We also see the estimation error variance goes to zero as time progresses (but the prediction error variance doesn't go to zero because the process noise is not vanishing like the measurement noise).

In part (b), we have an example output shown in Figure 4, the Kalman gain shown in Figure 5, and the MSE of the predictions and estimates shown in Figure 6. In this case the measurement noise is stationary (i.i.d.), so the Kalman gain quickly converges to a value between zero and one to balance the prediction and the innovation. We can also confirm our results are correct in part (b) by using our results from Problem 2 above. Problem 2 says our steady-state prediction error should go to

$$p = \frac{(-r' + f^2 r' + q') + \sqrt{(-r' + f^2 r' + q')^2 + 4q'r'}}{2}$$

where $r' = r/h^2 = 1$, $f = a = 0.9$, and $q' = \sigma_u^2 = 1$. Plugging these values in, we have $p = 1.4839$. Similarly, problem 2 says our steady-state estimation error should go to

$$s = \frac{(-r' + f^2 r' - q') + \sqrt{(-r' + f^2 r' - q')^2 + 4f^2 q'r'}}{2f^2}.$$

and plugging in the values for part (b) gives us $s = 0.5974$. Both of these values agree with what we see in Figure 6, which is nice.

In part (c), we have an example output shown in Figure 7, the Kalman gain shown in Figure 8, and the MSE of the predictions and estimates shown in Figure 9. In this case, the measurement noise is blowing up as time progresses and we see that the Kalman gain is going to zero because new observations are becoming less reliable. We also see that both the prediction and estimation MSEs are blowing up. Intuitively, this is because the state is moving around and we are getting fuzzier and fuzzier measurements of it.

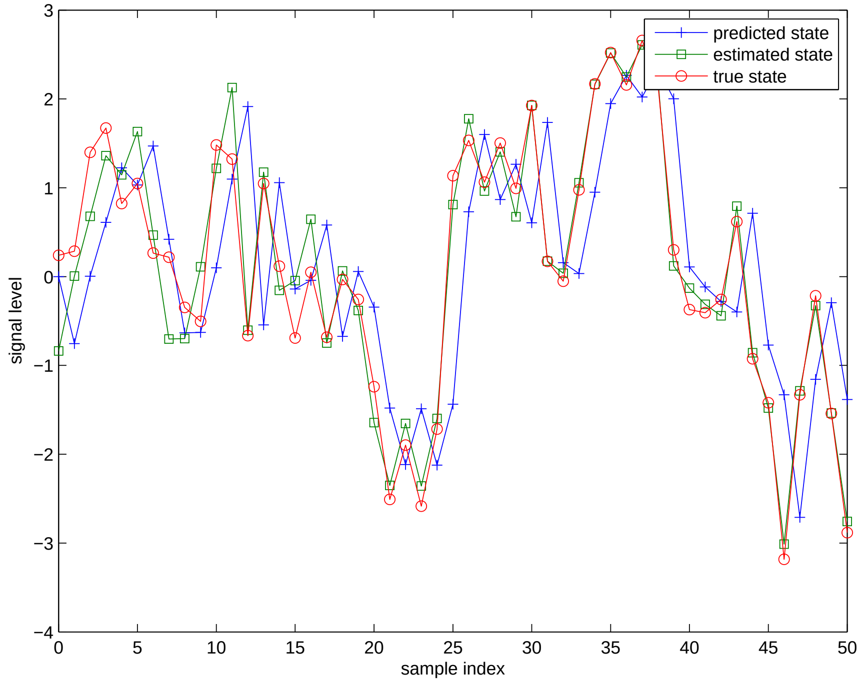

Figure 1: Example output for Kay 13.11 part (a)

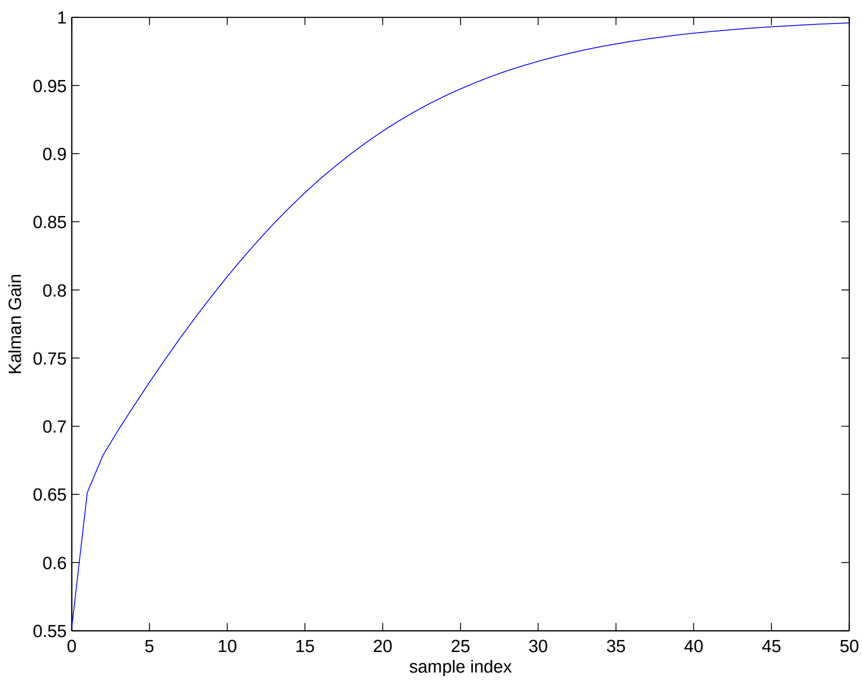

Figure 2: Kalman gain for Kay 13.11 part (a)

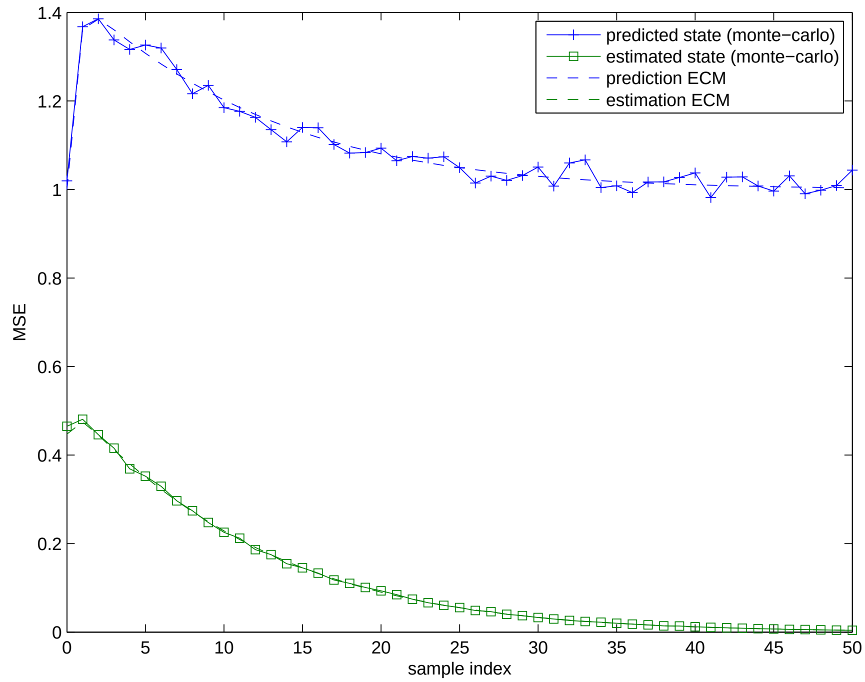

Figure 3: MSE for Kay 13.11 part (a)

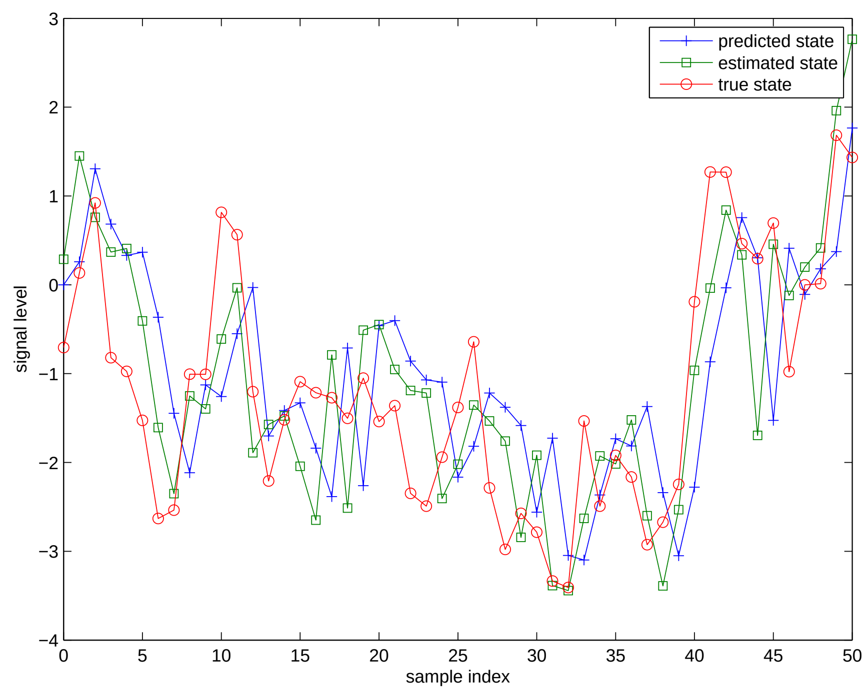

Figure 4: Example output for Kay 13.11 part (b)

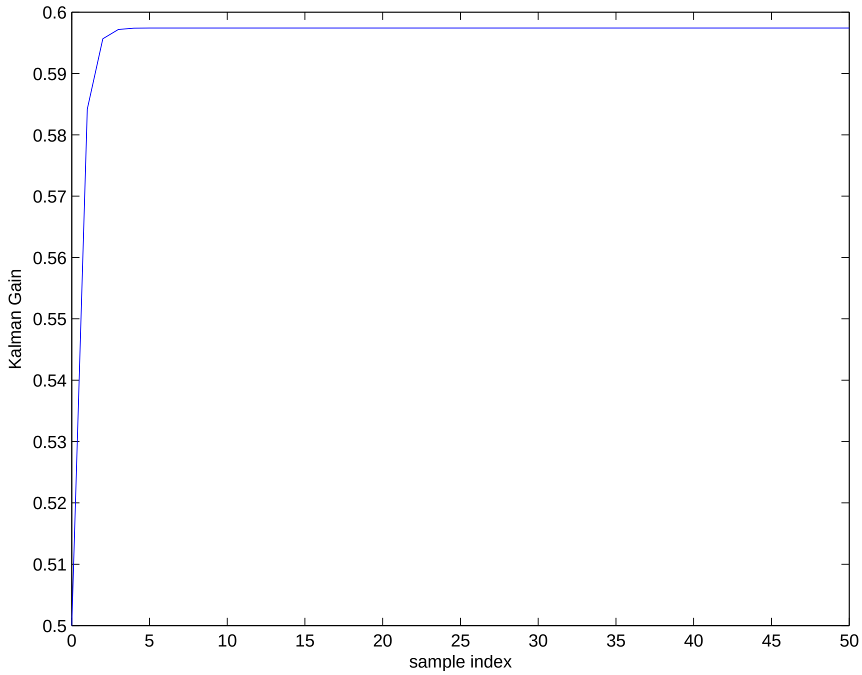

Figure 5: Kalman gain for Kay 13.11 part (b)

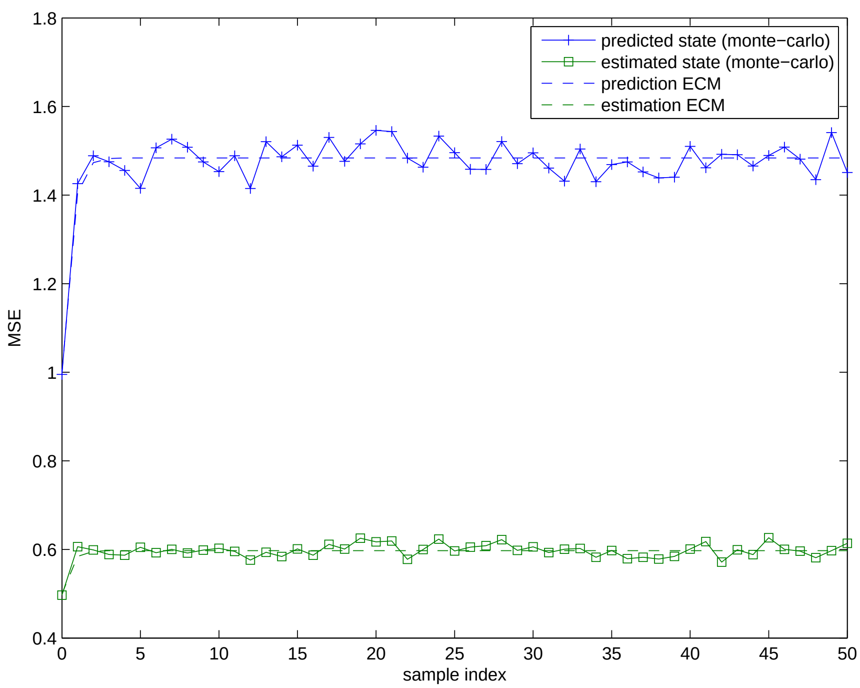

Figure 6: MSE for Kay 13.11 part (b)

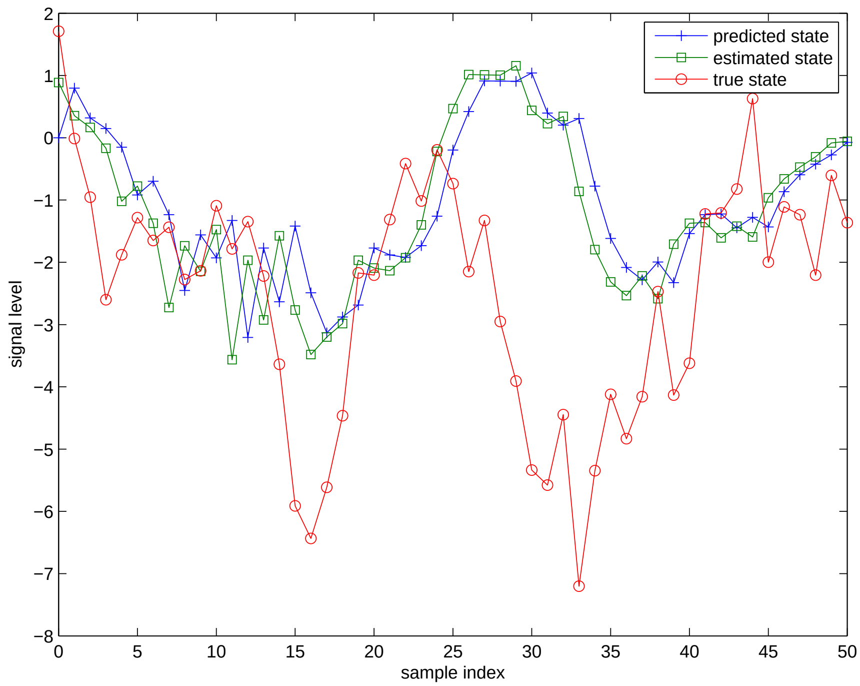

Figure 7: Example output for Kay 13.11 part (c)

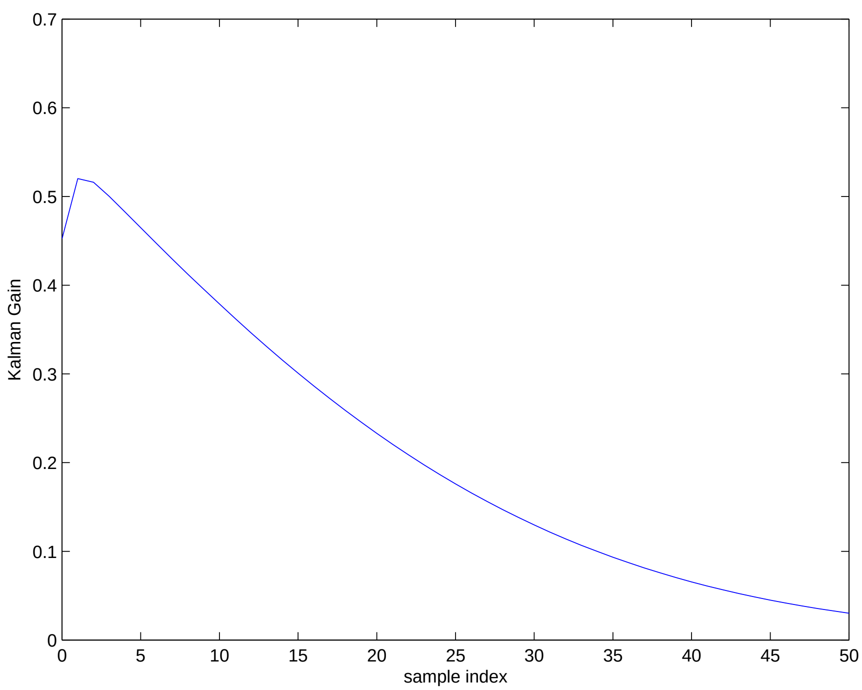

Figure 8: Kalman gain for Kay 13.11 part (c)

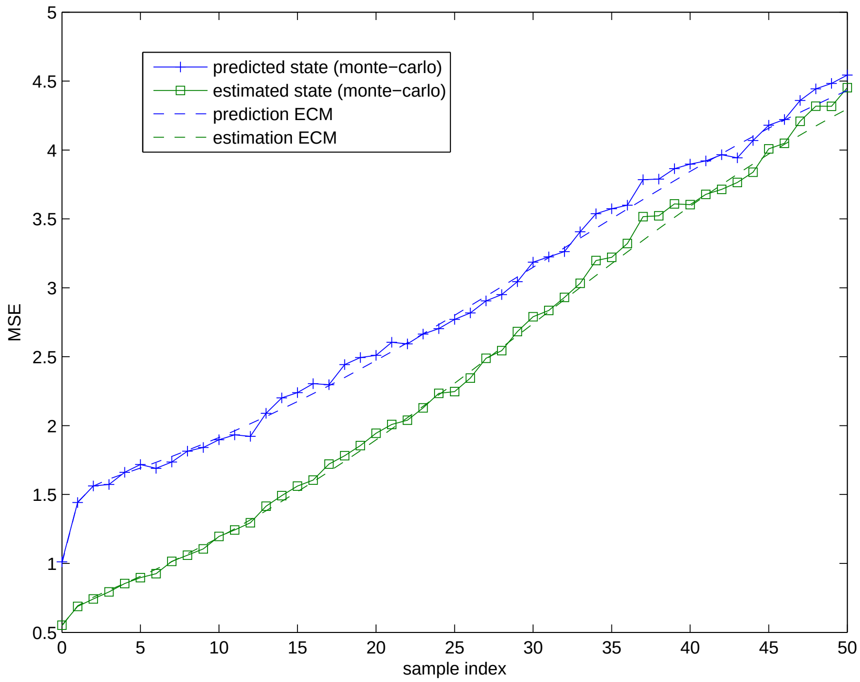

Figure 9: MSE for Kay 13.11 part (c)