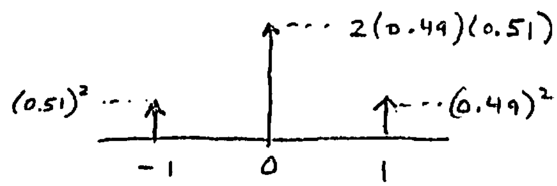
1. Let X be the number of shots made by team A.
Let Y be the " " " " " B.

Let $A_i = \begin{cases} 1 & \text{if team A makes } i\text{th shot} \\ 0 & \text{if team A misses } i\text{th shot} \end{cases}$

B_i defined similarly

Let $C_i = A_i - B_i = \begin{cases} 1 & \text{team A makes } i\text{th shot, B misses } i\text{th shot} \\ 0 & \text{both make or both miss} \\ -1 & \text{A misses } i\text{th shot, B makes } i\text{th shot} \end{cases}$

Note that C_i is a random variable with the following pdf



$E[C_i] = -0.02$

$\text{var}[C_i] = 0.4998$

Let $Z = \sum_{i=1}^{100} C_i$ $\qquad E[Z] = -2$; $\text{var}[Z] = 49.98$

$\text{Prob}[\text{team A wins}] = \text{Prob}[Z > 0] = \sum_{i=1}^{100} \text{Prob}[Z = i]$ ← difficult

Since we are adding up 100 of these C_i random variables, each with finite mean and variance, we can use the central limit theorem (CLT)...

Let $\bar{Z} = \dfrac{Z+2}{\sqrt{49.98}}$ ← this is a zero mean and unit variance RV

$$P(Z>0) = P\left[\sqrt{49.98}\,\bar{Z} - 2 > 0\right] = P\left[\bar{Z} > \frac{2}{\sqrt{49.98}}\right]$$

$$= P\left[\bar{Z} > 0.2829\right]$$

continued...

problem 1 continued.

Since $\bar{Z}$ is approximately distributed as $N(0,1)$ by the CLT, we can compute

$$P[\bar{Z} > 0.2829] = Q(0.2829) \approx 0.3886$$

Hence, the probability of team A winning is about 0.3886

Computer simulation shows probability A wins ≈ 0.363

So the CLT gives a good approximation here.

```
% ---------------------------------------------------------
% ECE531 Spring 2011 HW1 Problem 1
% computer simulation of 100 shot basketball game
% ---------------------------------------------------------
% Parameters
% ---------------------------------------------------------
PA = 0.49;          % probability team A makes a shot
PB = 0.51;          % probability team B makes a shot
N = 100;            % number of shots
iterations = 1e5;   % number of iterations
% ---------------------------------------------------------
Awins = zeros(1,iterations);
for i=1:iterations,
    A = rand(1,N)<PA;
    B = rand(1,N)<PB;
    X = sum(A);
    Y = sum(B);
    Z = X-Y;
    if Z>0
        Awins(i) = 1;
    end
end
disp('Probability A wins:');
sum(Awins)/iterations
```

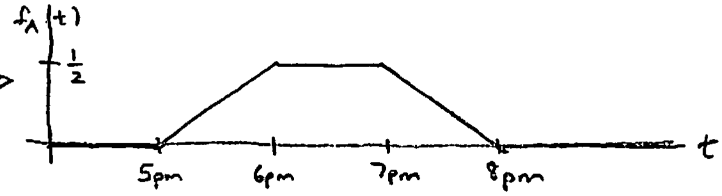
2. a) Let the time of arrival in Boston be A; Then

$$f_A(t) = f_{T_1}(t) * f_\tau(t)$$

↑ convolution

$T_1 \sim U(4pm, 6pm)$

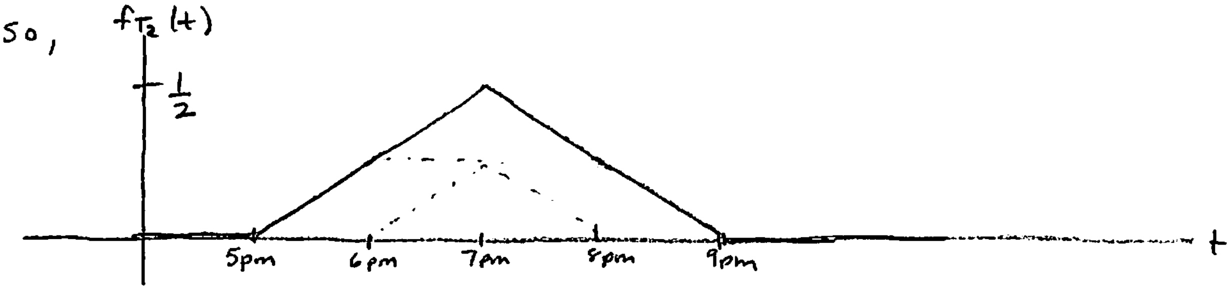
$\tau \sim U(1hr, 2hr)$ $\Rightarrow$



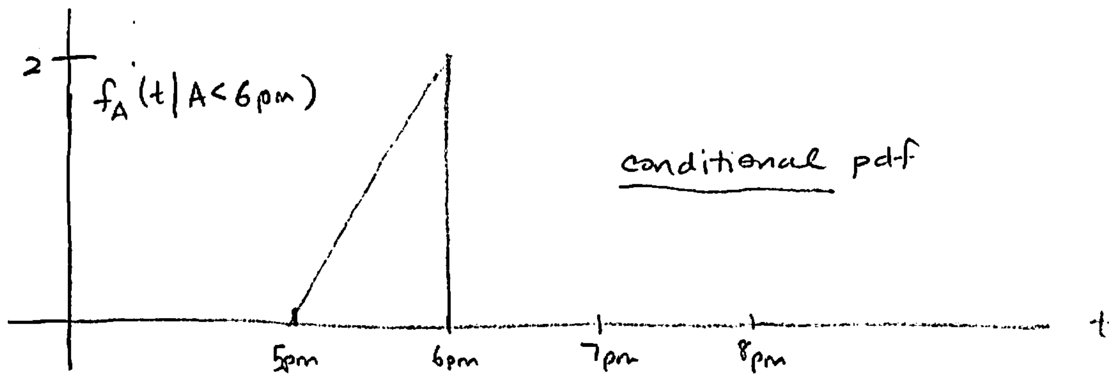
Now, the time at which you get a taxi is

$f_{T_2}(t) = f_A(t) * f_W(t)$ where $f_W(t) = \frac{1}{2}\delta(t) + \frac{1}{2}\delta(t-1)$

So,



b) Now we must condition on the event that we arrived in Boston before 6pm.

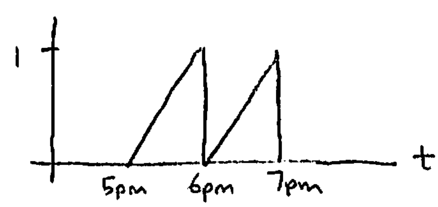


<u>conditional</u> pdf

Let W be the waiting time for a taxi.

convolution again
↓

Hence $f_{T_2}(t|A<6pm) = f_A(t|A<6pm) * f_W(t)$



and $E[T_2|A<6pm] = \int_{-\infty}^{\infty} t\, f_{T_2}(t|A<6pm)\,dt$

continued . .

$$\int_{-\infty}^{\infty} t\, f_{T_2}(t \mid A < 6pm)\, dt = \int_5^6 t(t-5)\, dt + \int_6^7 t(t-6)\, dt = \frac{37}{6}$$

Hence, conditioned on the event that we arrived in Boston before 6 pm, the expected value of the time at which we get a taxi is

$$E[T_2 \mid A < 6pm] = \frac{37}{6} = \boxed{6:10 \text{ pm}}$$

3. a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$$

Plug in f_{XY}, solve for A, $\boxed{A = 1}$

b) $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy = \int_0^1 (x+y)\, dy = \boxed{\begin{cases} x + \frac{1}{2} & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}}$$

c) $$P[X > 0.5 \mid Y > 0.5] = \frac{P[X > 0.5, Y > 0.5]}{P[Y > 0.5]}$$

$$P[X > 0.5, Y > 0.5] = \int_{0.5}^{1}\int_{0.5}^{1} (x+y)\, dx\, dy = \frac{3}{8}$$

$$P[Y > 0.5] = \int_{0.5}^{1}\int_{0}^{1} (x+y)\, dx\, dy = \frac{5}{8}$$

hence $$\boxed{P[X > 0.5 \mid Y > 0.5] = \frac{3}{5}}$$

4. Kay 1.1

$$\text{Prob}\left[x[0] > \tfrac{1}{2} \mid \mathcal{H}_0\right] = \text{Prob}\left[w[0] > \tfrac{1}{2}\right]$$

$w[0] \sim \mathcal{N}(0, \sigma^2)$, let $v = \frac{w[0]}{\sigma}$ then $v \sim \mathcal{N}(0,1)$

hence $\text{Prob}\left[w[0] > \tfrac{1}{2}\right] = \text{Prob}\left[\sigma v > \tfrac{1}{2}\right] = \text{Prob}\left[v > \tfrac{1}{2\sigma}\right]$

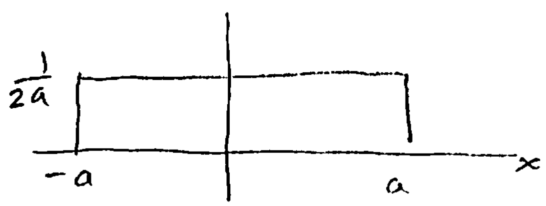
$$\boxed{= Q\left(\tfrac{1}{2\sigma}\right)}$$

When $\boxed{\sigma = 0.162}$, $Q\left(\frac{1}{2\sigma}\right) = 10^{-3}$ (you can use Matlab's erfcinv function to solve this)

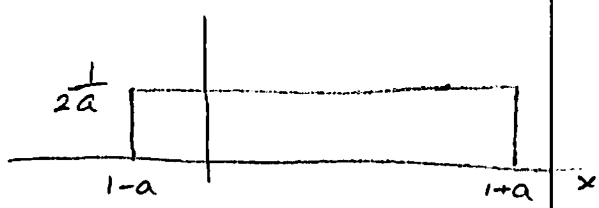
5. Decide $\begin{cases} \mathcal{H}_1 & \text{if } x[0] > \frac{1}{2} \\ \mathcal{H}_0 & \text{otherwise} \end{cases}$

$w[0]$ is uniformly distributed on $[-a, a]$

$f_{x[0]|\mathcal{H}_0}(x)$



$f_{x[0]|\mathcal{H}_1}(x)$



2 cases:

I. $0 < a \leq \frac{1}{2}$ $\Rightarrow$ We never make wrong decisions in this case because the noise can't cause $x[0] < \frac{1}{2}$ when $\mathcal{H}_1$ is true or $x[0] > \frac{1}{2}$ when $\mathcal{H}_0$ is true.

II. $a > \frac{1}{2}$

$$\text{Prob}\left[\text{decide } \mathcal{H}_1 \mid \mathcal{H}_0\right] = \text{Prob}\left[w[0] > \tfrac{1}{2}\right] = \int_{\frac{1}{2}}^{a} \frac{1}{2a}\, dx$$

$$= \frac{1}{2a}\left[a - \tfrac{1}{2}\right] = \frac{1}{2} - \frac{1}{4a}$$

Hence as a gets large, the probability of deciding $\mathcal{H}_1$ when the true state of nature is $\mathcal{H}_0$ goes to $\frac{1}{2}$.

continued...

problem 5 continued ...

similarly $\text{Prob}[\text{decide } \mathcal{H}_0 \mid \mathcal{H}_1] = \text{prob}[1+w[0] < \frac{1}{2}]$

$= \text{prob}[w[0] < -\frac{1}{2}] = \frac{1}{2} - \frac{1}{4a}$ by symmetry.

So, as a gets large, the probability of deciding $\mathcal{H}_0$ when the true state of nature is $\mathcal{H}_1$ also goes to $\frac{1}{2}$.

As a gets large, this detector is not very effective at distinguishing $\mathcal{H}_0$ and $\mathcal{H}_1$.

6. Kay 1.4

$P_e = \text{Prob}[x[0] > \frac{1}{2} \mid \mathcal{H}_0]\,\text{Prob}[\mathcal{H}_0] + \text{Prob}[x[0] < \frac{1}{2} \mid \mathcal{H}_1]\,\text{Prob}[\mathcal{H}_1]$

$\text{Prob}[\mathcal{H}_0] = \text{Prob}[\mathcal{H}_1] = \frac{1}{2}$

$\text{Prob}[x[0] > \frac{1}{2} \mid \mathcal{H}_0] = \text{Prob}[w[0] > \frac{1}{2}] = Q(\frac{1}{2\sigma})$

$\text{Prob}[x[0] < \frac{1}{2} \mid \mathcal{H}_1] = \text{Prob}[w[0] < -\frac{1}{2}] = Q(\frac{1}{2\sigma})$

so

$P_e = Q(\frac{1}{2\sigma})$ → plot on next page

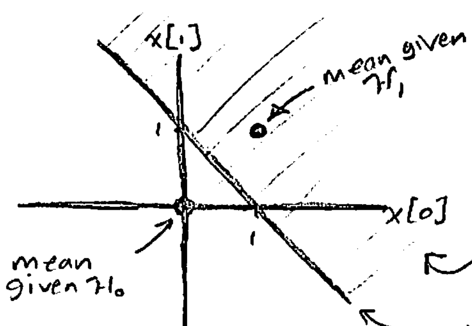
For small σ, $Q(\frac{1}{2\sigma}) \to 0$ and P_e is small. This makes sense because the variance of the noise is small in this case.

For large σ, $Q(\frac{1}{2\sigma}) \to \frac{1}{2}$ and $P_e \to \frac{1}{2}$. This makes sense because we make errors 50% of the time when the noise has large variance.

7. Kay 1.5

$E\left\{\begin{bmatrix} x[0] \\ x[1] \end{bmatrix} \Big| \mathcal{H}_0\right\} = E\left\{\begin{bmatrix} w[0] \\ w[1] \end{bmatrix}\right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

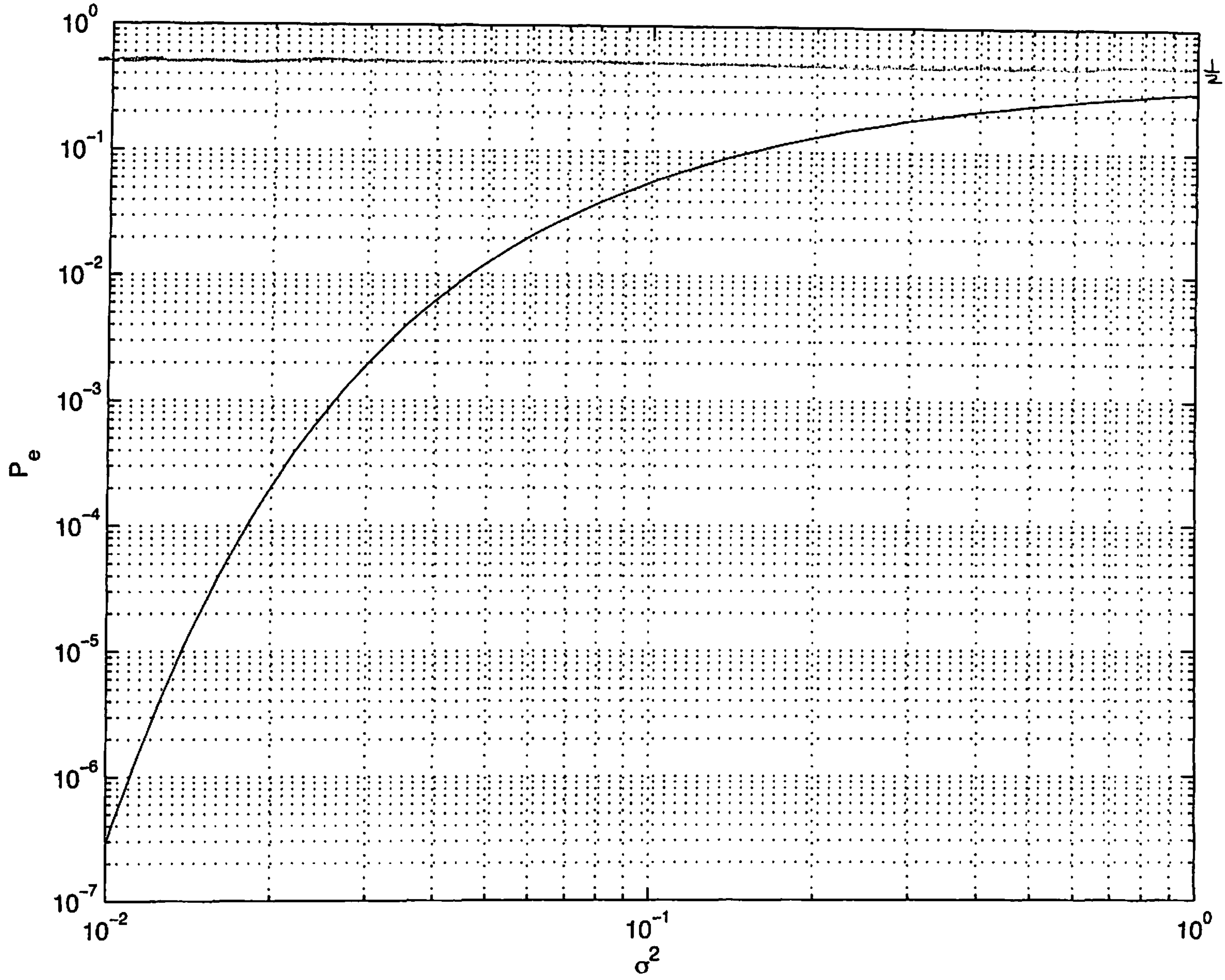
$E\left\{\begin{bmatrix} x[0] \\ x[1] \end{bmatrix} \Big| \mathcal{H}_1\right\} = E\left\{\begin{matrix} 1+w[0] \\ 1+w[1] \end{matrix}\right\} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



the shaded region is where we decide $\mathcal{H}_1$, otherwise we decide $\mathcal{H}_0$.

this line bisects the line between $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This is a <u>minimum distance detector</u> in the sense that, when you observe $x[0]$ and $x[1]$, you decide $\mathcal{H}_0$ if the observation is closer to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or you decide $\mathcal{H}_1$ if the observation is closer to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.


10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
P_e
$\frac{1}{2}$
10^{-2}
10^{-1}
10^0
σ^2