ECE531 Homework Assignment Number 2

Due by 8:50pm on Wednesday 2-Feb-2011

Make sure your reasoning and work are clear to receive full credit for each problem.

1. 14 points. Suppose you work in a microprocessor manufacturing facility and that, before boxing and shipping, each microprocessor undergoes a quality check to avoid shipping defective units. The quality checking machine has the following characteristics:

   - It declares good microprocessors to be defective (D) with probability $p = 0.15$.
   - It declares defective microprocessors to be good (G) with probability $q = 0.03$.

   Suppose there are n such quality checking machines that give independent results with the same probabilities. Also let $\mathcal{H}_0$ be the hypothesis that the microprocessor is good and let $\mathcal{H}_1$ be the hypothesis that the microprocessor is defective.

   (a) 2 points. For the case $n = 1$ (each microprocessor is only checked by one machine), explicitly describe the states $\mathcal{X}$, possible observations $\mathcal{Y}$, the conditional probability matrix P, and the set of all possible deterministic decision rules. What type of hypothesis testing problem is this?

   Solution:

   - States $\mathcal{X} = \{x_0, x_1\}$ with x_0 being the state that the microprocessor is good and x_1 being the state that the microprocessor is defective ($N = 2$).
   - Possible observations for one quality check are $\mathcal{Y} = \{y_0, y_1\}$ with $y_0 = \mathsf{G}$ being the observation that the quality checking machine indicates a "good" microprocessor and $y_1 = \mathsf{D}$ being the observation that the quality checking machine indicates a "bad" microprocessor. ($L = 2$)
   - The conditional probability matrix P can be derived from the problem description as

   $$P = \begin{bmatrix} P_{x=x_0}(y = y_0) & P_{x=x_1}(y = y_0) \\ P_{x=x_0}(y = y_1) & P_{x=x_1}(y = y_1) \end{bmatrix} = \begin{bmatrix} 0.85 & 0.03 \\ 0.15 & 0.97 \end{bmatrix} \tag{1}$$

   - There are only $M^L = 2^2 = 4$ possible deterministic decision rules in this case. The set of deterministic decision matrices is

   $$D \in \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\} \tag{2}$$

   - This is a **simple**, **binary** hypothesis testing problem.

   (b) 2 points. Assume the uniform cost assignment (UCA) and $n = 1$. For each possible deterministic decision rule, plot the conditional risk vector $R(D) = [R_0(D), R_1(D)]^\top$ in the R_0, R_1 plane. Color in the set of all achievable conditional risk vectors in the R_0, R_1 plane and also identify the optimal tradeoff surface.

   Solution:

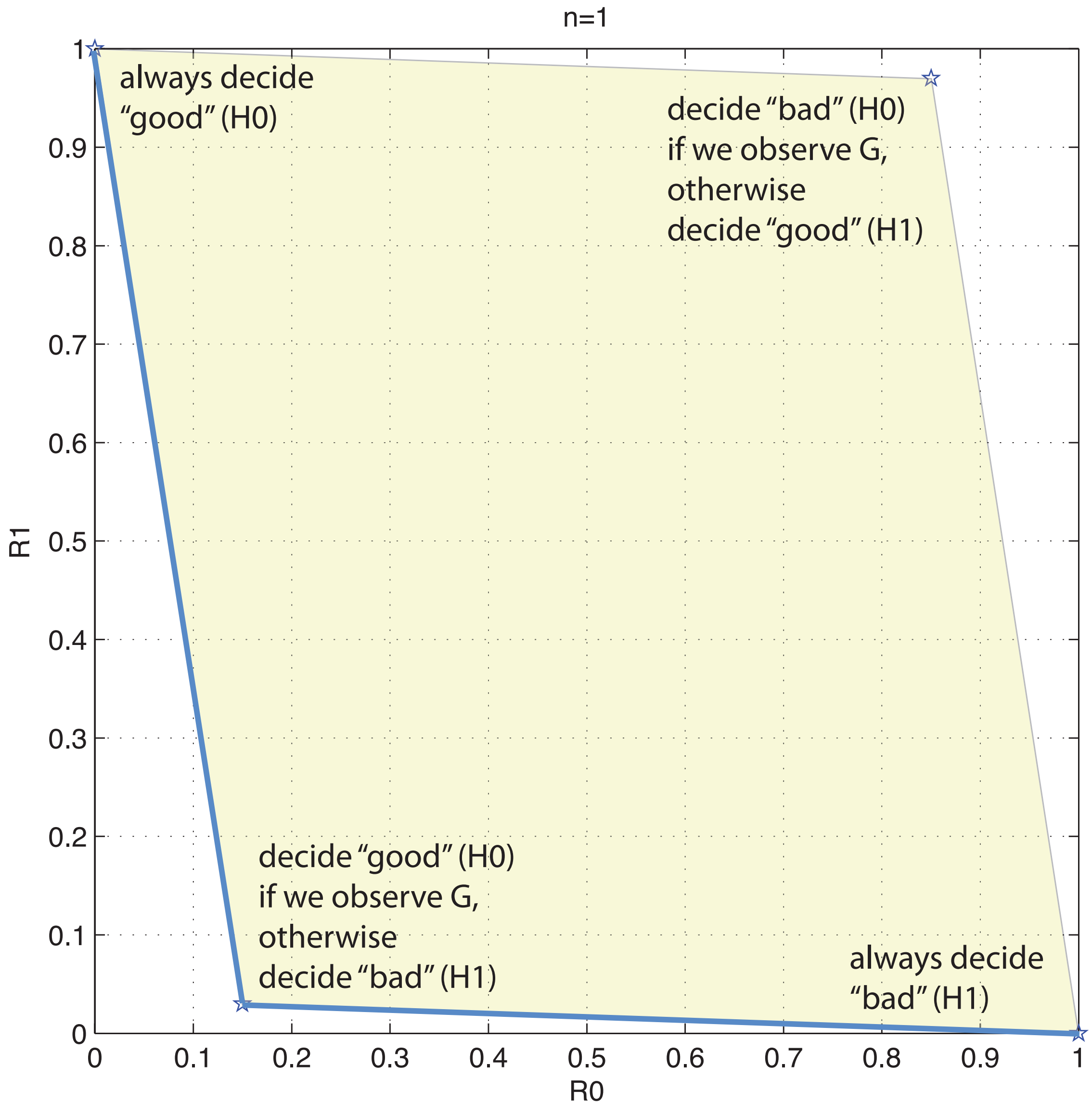

Figure 1: The set of all achievable CRVs is shown in yellow. The pentagrams are the CRVs associated with the deterministic decision rules. The thicker blue line is the optimal tradeoff surface.

(c) 2 points. Determine the Neyman-Pearson decision rule for significance levels $\alpha = 0.03$ and $\alpha = 0.01$ when $n = 1$. What is the probability of detection, i.e. β, for the N-P detector at each of these significance levels?

Solution: Note that a "false positive" here is when we declare the microprocessor to be defective when it is actually good. We have a deterministic decision rule that gives a false positive probability of 0.15 and another deterministic decision rule that gives a false positive probability of 0, but none that give the desired false positive probabilities of 0.03 and 0.01. So, first form the likelihood ratio vector

$$L = \begin{bmatrix} 0.035 \\ 6.467 \end{bmatrix} \tag{3}$$

All N-P detectors will be of the form

$$\rho^{NP}(y_\ell) = \begin{cases} 1 & \text{if } L_\ell > v \\ \gamma & \text{if } L_\ell = v \\ 0 & \text{if } L_\ell < v \end{cases} \tag{4}$$

- Inspection of Figure 1 reveals that when $\alpha = 0.03$, we will need a randomized decision rule. We achieve the desired probability of false positive $P_{\mathsf{fp}} = \alpha = 0.03$ when $v = 6.467$ and

$$\gamma = \frac{\alpha - \sum_{\ell:L_\ell>v} P_{\ell,0}}{\sum_{\ell:L_\ell=v} P_{\ell,0}} = \frac{0.03 - 0}{0.15} = 0.2. \tag{5}$$

  In other words, if we observe G, we decide the microprocessor is good. If we observe D, we decide the microprocessor is defective with probability 0.2, otherwise we decide the microprocessor is good. The probability of detection in this case is $\beta = 0.2 \times 0.97 = 0.194$. This can be confirmed by looking at Figure 1.

- When $\alpha = 0.01$, the same basic analysis applies. To achieve the desired probability of false positive $P_{\mathsf{f}p} = \alpha = 0.01$, we set $v = 6.467$ and

$$\gamma = \frac{\alpha - \sum_{\ell:L_\ell>v} P_{\ell,0}}{\sum_{\ell:L_\ell=v} P_{\ell,0}} = \frac{0.01 - 0}{0.15} = 0.0667 \tag{6}$$

  In other words, if we observe G, we decide the microprocessor is good. If we observe D, we decide the microprocessor is defective with probability 0.0667, otherwise we decide the microprocessor is good. The probability of detection in this case is $\beta = 0.0667 \times 0.97 = 0.0647$. This can also be confirmed by looking at Figure 1.

(d) 6 points. Repeat (a)-(c) for the case $n = 2$ (each microprocessor is checked by two separate machines). Discuss on the how the set of achievable conditional risk vectors and the optimal tradeoff surface change when two quality checks are performed.

Solution: (a)

- States are unchanged.
- Let's declare our observations to be the number of D's that we observe. Then the possible observations for two quality checks are $\mathcal{Y} = \{y_0, y_1, y_2\}$ with $y_\ell = \ell$. Note that $L = 3$ now.
- The conditional probability matrix P can be derived from the problem description as

$$P = \begin{bmatrix} P_{x=x_0}(y=y_0) & P_{x=x_1}(y=y_0) \\ P_{x=x_0}(y=y_1) & P_{x=x_1}(y=y_1) \\ P_{x=x_0}(y=y_2) & P_{x=x_1}(y=y_2) \end{bmatrix} = \begin{bmatrix} 0.723 & 0.0009 \\ 0.255 & 0.0582 \\ 0.022 & 0.9409 \end{bmatrix} \tag{7}$$

- There are $M^L = 2^3 = 8$ possible deterministic decision rules in this case. The set of deterministic decision matrices is

$$D \in \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\} \tag{8}$$

- This is still a **simple**, **binary** hypothesis testing problem.

Solution: (b)

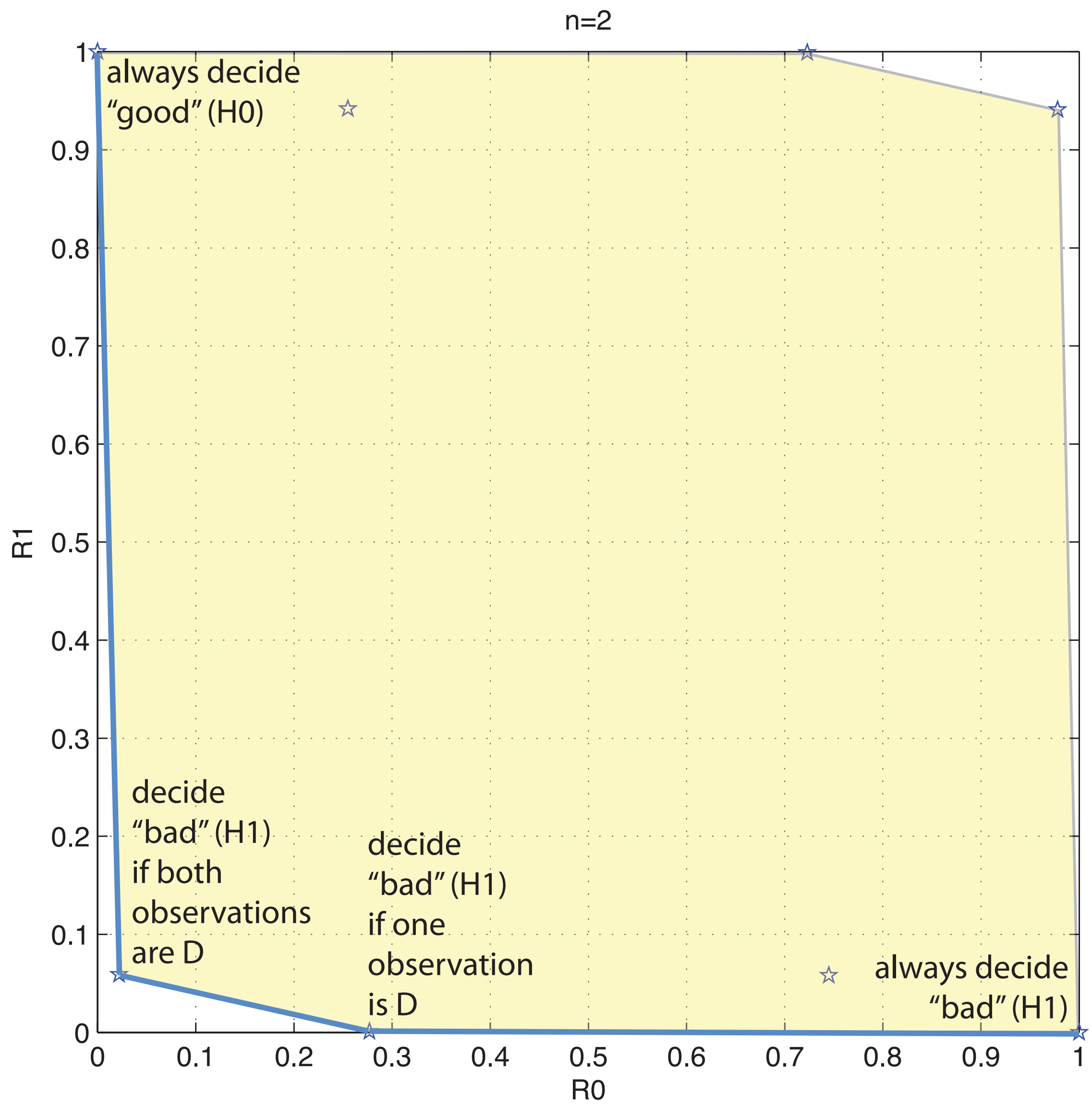

Figure 2: The set of all achievable CRVs is shown in yellow. The pentagrams are the CRVs associated with the deterministic decision rules. The thicker blue line is the optimal tradeoff surface.

Solution: (c)

First form the likelihood ratio vector

$$L = \begin{bmatrix} 0.0012 \\ 0.2282 \\ 42.7682 \end{bmatrix} \tag{9}$$

All N-P detectors will be of the form

$$\rho^{NP}(y_\ell) = \begin{cases} 1 & \text{if } L_\ell > v \\ \gamma & \text{if } L_\ell = v \\ 0 & \text{if } L_\ell < v \end{cases} \tag{10}$$

- We achieve the desired probability of false positive $P_{\mathsf{fp}} = \alpha = 0.03$ when $v = 0.2282$ and

$$\gamma = \frac{\alpha - \sum_{\ell:L_\ell > v} P_{\ell,0}}{\sum_{\ell:L_\ell = v} P_{\ell,0}} = \frac{0.03 - 0.022}{0.255} = 0.035. \tag{11}$$

  In other words, if we observe two D's, we decide the microprocessor is defective. If we observe one D, we decide the microprocessor is defective with probability 0.035, otherwise we decide the microprocessor is good. If we observe no D's, we decide the microprocessor is good. The probability of detection in this case is $\beta = 0.9409 + 0.035 \times 0.0583 = 0.9429$. This can be confirmed by looking at Figure 2.

- When $\alpha = 0.01$, the same basic analysis applies. To achieve the desired probability of false positive $P_{\mathsf{f}p} = \alpha = 0.01$, we set $v = 42.7682$ and

$$\gamma = \frac{\alpha - \sum_{\ell:L_\ell > v} P_{\ell,0}}{\sum_{\ell:L_\ell = v} P_{\ell,0}} = \frac{0.01 - 0}{0.022} = 0.4545. \tag{12}$$

  In other words, if we observe two D's, we decide the microprocessor is defective. with probability 0.4545, otherwise we decide the microprocessor is good. If we observe one or no D's, we decide the microprocessor is good. The probability of detection in this case is $\beta = 0.9409 \times 0.4545 = 0.4276$. This can be confirmed by looking at Figure 2.

(e) 2 points. How many quality checking machines would be needed to satisfy $P_{\mathsf{fp}} < 0.001$ and $P_D > 0.999$?

Solution: You can get pretty close at $n = 7$, but you need $n = 8$ independent quality checking machines (and a randomized decision rule) to get $P_{\mathsf{fp}} = R_0(D) < 0.001$ and $P_D > 0.999 \Leftrightarrow R_1(D) < 0.001$. See Figure 3.

Here is my code:

```
%--------------------------------------------------------------
% ECE531 Spring 2011
% DRB 28-Jan-2011
% Solution to Homework 2 Problem 1
% Quality Checker
% The observation is the number of D's observed (y in 0,1,...,n)
%--------------------------------------------------------------
% USER PARAMETERS BELOW
%--------------------------------------------------------------
n = 8;          % number of independent quality checking machines
p = 0.15;       % prob observing D given good microprocessor (x0/H0)
q = 0.03;       % prob observing G given defective microprocessor (x1/H1)
C = [0 1 ; 1 0]; % UCA
N = 2;          % number of hypotheses
M = 2;          % number of states
%--------------------------------------------------------------

L = n+1;              % number of possible observations
totD = M^L;           % total number of decision matrices
B = makebinary(L,1);

% make conditional probability matrix
P0 = zeros(L,1);              % first column
P1 = zeros(L,1);              % second column
for i = 0:(L-1),
    P0(i+1) = nchoosek(n,i) * p^i * (1-p)^(n-i);
    P1(i+1) = nchoosek(n,i) * (1-q)^i * q^(n-i);
end
P = [P0 P1];

% compute CRVs for all possible deterministic decision matrices
for i = 0:(totD-1),
    D = [ B(:,i+1)' ; 1-B(:,i+1)' ];     % decision matrix
    for j=0:N-1,
        R(j+1,i+1) = C(:,j+1)'*D*P(:,j+1);
    end
end

% plot
plot(R(1,:),R(2,:),'p'); xlabel('R0'); ylabel('R1');
axis([0 5e-3 0 5e-3]);
axis square; grid on
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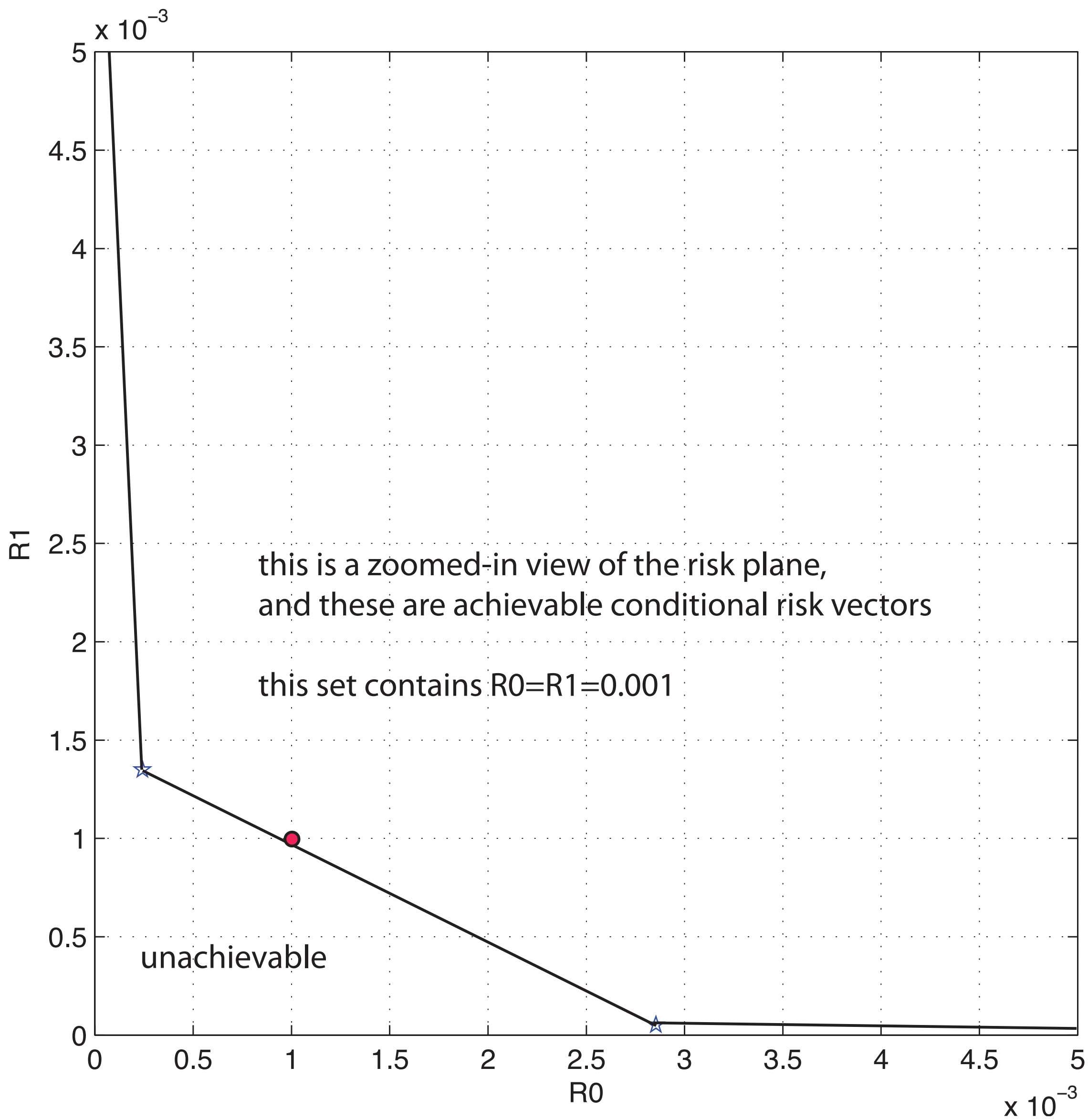

Figure 3: Zoomed in view of R_0, R_1 plane showing $R_0 = R_1 = 0.001$ is achievable when $n = 8$.

2. 5 points. Suppose you want to determine if a six-sided die is fair by rolling it and observing the outcomes of the rolls. There are two hypotheses

   - $\mathcal{H}_0$ the die is fair (all six numbers are equiprobable)
   - $\mathcal{H}_1$ the die in unfair with the following probabilities: $P(1) = 0.27$, $P(2) = 0.10$, $P(3) = 0.05$, $P(4) = 0.20$, $P(5) = 0.16$, and $P(6) = 0.22$.

   You observe only one roll of the die. Determine the Neyman-Pearson decision rule for significance level α and plot the probability of detection β as a function of α.

   Solution: Form the conditional probability matrix P and the likelihood ratio vector L:

$$P = \begin{bmatrix} 1/6 & 0.27 \\ 1/6 & 0.10 \\ 1/6 & 0.05 \\ 1/6 & 0.20 \\ 1/6 & 0.16 \\ 1/6 & 0.22 \end{bmatrix} \qquad L = \begin{bmatrix} 1.62 \\ 0.60 \\ 0.30 \\ 1.20 \\ 0.96 \\ 1.32 \end{bmatrix} \tag{13}$$

   All N-P detectors will be of the form

$$\rho^{NP}(y_\ell) = \begin{cases} 1 & \text{if } L_\ell > v \\ \gamma & \text{if } L_\ell = v \\ 0 & \text{if } L_\ell < v \end{cases} \tag{14}$$

   The optimal N-P decision rule depends on the significance level α as follows

   - $\alpha = 0$: Always decide $\mathcal{H}_0$, i.e. $v = 1.62$ and $\gamma = 0$.
   - $0 < \alpha \leq 1/6$: $v = 1.62$ and

$$\gamma = \frac{\alpha - \sum_{\ell:L_\ell>v} P_{\ell,0}}{\sum_{\ell:L_\ell=v} P_{\ell,0}} = \frac{\alpha - 0}{1/6} = 6\alpha. \tag{15}$$

     In other words, if the outcome of the die roll is a one, we decide $\mathcal{H}_1$ with probability 6α, otherwise we decide $\mathcal{H}_0$. If the outcome of the die roll is any number other than one, we decide $\mathcal{H}_0$. The probability of detection in this case is $\beta = 0.27 \times 6\alpha = 1.62\alpha$.
   - $1/6 < \alpha \leq 2/6$: $v = 1.32$ and

$$\gamma = \frac{\alpha - \sum_{\ell:L_\ell>v} P_{\ell,0}}{\sum_{\ell:L_\ell=v} P_{\ell,0}} = \frac{\alpha - 1/6}{1/6} = 6\alpha - 1. \tag{16}$$

     In other words, if the outcome of the die roll is a one, we decide $\mathcal{H}_1$. If the outcome of the die roll is a six, we decide $\mathcal{H}_1$ with probability $6\alpha - 1$, otherwise we decide $\mathcal{H}_0$. If the outcome of the die roll is any number other than one or six, we decide $\mathcal{H}_0$. The probability of detection in this case is $\beta = 0.27 + 0.22 \times (6\alpha - 1) = 1.32\alpha + 0.05$.
   - $2/6 < \alpha \leq 3/6$: $v = 1.20$ and

$$\gamma = \frac{\alpha - \sum_{\ell:L_\ell>v} P_{\ell,0}}{\sum_{\ell:L_\ell=v} P_{\ell,0}} = \frac{\alpha - 2/6}{1/6} = 6\alpha - 2. \tag{17}$$

     In other words, if the outcome of the die roll is a one or six, we decide $\mathcal{H}_1$. If the outcome of the die roll is a four, we decide $\mathcal{H}_1$ with probability $6\alpha - 2$, otherwise we decide $\mathcal{H}_0$. If the outcome of the die roll is any number other than one, four, or six, we decide $\mathcal{H}_0$. The probability of detection in this case is $\beta = 0.27 + 0.22 + 0.20 \times (6\alpha - 2) = 1.20\alpha + 0.09$.

- $3/6 < \alpha \leq 4/6$: $v = 0.96$ and

$$\gamma = \frac{\alpha - \sum_{\ell: L_\ell > v} P_{\ell,0}}{\sum_{\ell: L_\ell = v} P_{\ell,0}} = \frac{\alpha - 3/6}{1/6} = 6\alpha - 3. \tag{18}$$

  In other words, if the outcome of the die roll is a one, four, or six, we decide $\mathcal{H}_1$. If the outcome of the die roll is a five, we decide $\mathcal{H}_1$ with probability $6\alpha - 3$, otherwise we decide $\mathcal{H}_0$. If the outcome of the die roll is a two or a three, we decide $\mathcal{H}_0$. The probability of detection in this case is $\beta = 0.27 + 0.22 + 0.20 + 0.16 \times (6\alpha - 3) = 0.96\alpha + 0.21$.

- $4/6 < \alpha \leq 5/6$: $v = 0.60$ and

$$\gamma = \frac{\alpha - \sum_{\ell: L_\ell > v} P_{\ell,0}}{\sum_{\ell: L_\ell = v} P_{\ell,0}} = \frac{\alpha - 4/6}{1/6} = 6\alpha - 4. \tag{19}$$

  In other words, if the outcome of the die roll is a one, four, five, or six, we decide $\mathcal{H}_1$. If the outcome of the die roll is a two, we decide $\mathcal{H}_1$ with probability $6\alpha - 4$, otherwise we decide $\mathcal{H}_0$. If the outcome of the die roll is a three, we decide $\mathcal{H}_0$. The probability of detection in this case is $\beta = 0.27 + 0.22 + 0.20 + 0.16 + 0.10 \times (6\alpha - 4) = 0.60\alpha + 0.45$.

- $5/6 < \alpha < 6/6$: $v = 0.30$ and

$$\gamma = \frac{\alpha - \sum_{\ell: L_\ell > v} P_{\ell,0}}{\sum_{\ell: L_\ell = v} P_{\ell,0}} = \frac{\alpha - 5/6}{1/6} = 6\alpha - 5. \tag{20}$$

  In other words, if the outcome of the die roll is anything but a three, we decide $\mathcal{H}_1$. If the outcome of the die roll is a three, we decide $\mathcal{H}_1$ with probability $6\alpha - 5$, otherwise we decide $\mathcal{H}_0$. The probability of detection in this case is $\beta = 0.27 + 0.22 + 0.20 + 0.16 + 0.10 + 0.05 \times (6\alpha - 5) = 0.30\alpha + 0.70$.

- $\alpha = 1$. Always decide $\mathcal{H}_1$, i.e. $v = 0.30$ and $\gamma = 1$.

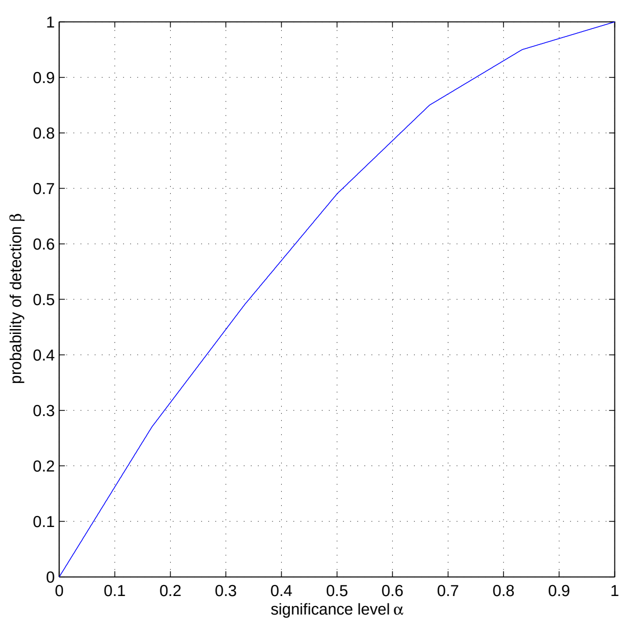

Figure 4: Probability of detection of an unfair die as a function of the significance level of the test.

3. 3 points. Kay II:3.1.

Solution: Kay uses $x[0]$ as the observation whereas our notation is y. I will use our notation for the following solution. The N-P detector will be of the form

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } L(y) \geq v \\ 0 & \text{if } L(y) < v. \end{cases} \tag{21}$$

Randomization is unnecessary here because the conditional observation distributions are continuous (not discrete). The probability of $L(y) = v$ is zero for all v. The likelihood ratio

$$L(y) = \frac{p_{x=x_1}(y)}{p_{x=x_0}(y)} = \exp\left\{y - \frac{1}{2}\right\} \tag{22}$$

It is convenient in this case to write the decision rule in terms of the *log*-likelihood ratios, i.e.

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } \ln(L(y)) \geq \ln(v) \\ 0 & \text{if } \ln(L(y)) < \ln(v). \end{cases} \tag{23}$$

Let $v' = \ln v + 1/2$. Then the N-P decision rule is

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } y \geq v' \\ 0 & \text{if } y < v'. \end{cases} \tag{24}$$

where v' is chosen to satisfy the false positive probability constraint.

The probability of a false positive is

$$P_{\mathsf{fp}} = \text{Prob}(y \geq v'\,;\, \text{state is } x_0) = Q(v') = \alpha \tag{25}$$

Hence $v' = Q^{-1}(\alpha)$. The probability of detection is

$$\beta = P_D = \text{Prob}(y \geq v'\,;\, \text{state is } x_1) = Q(v' - 1) = Q(Q^{-1}(\alpha) - 1) \tag{26}$$

Finally, the probability of a miss (false negative) is

$$P_M = \text{Prob}(y < v'\,;\, \text{state is } x_1) = 1 - Q(v' - 1) = 1 - Q(Q^{-1}(\alpha) - 1) \tag{27}$$

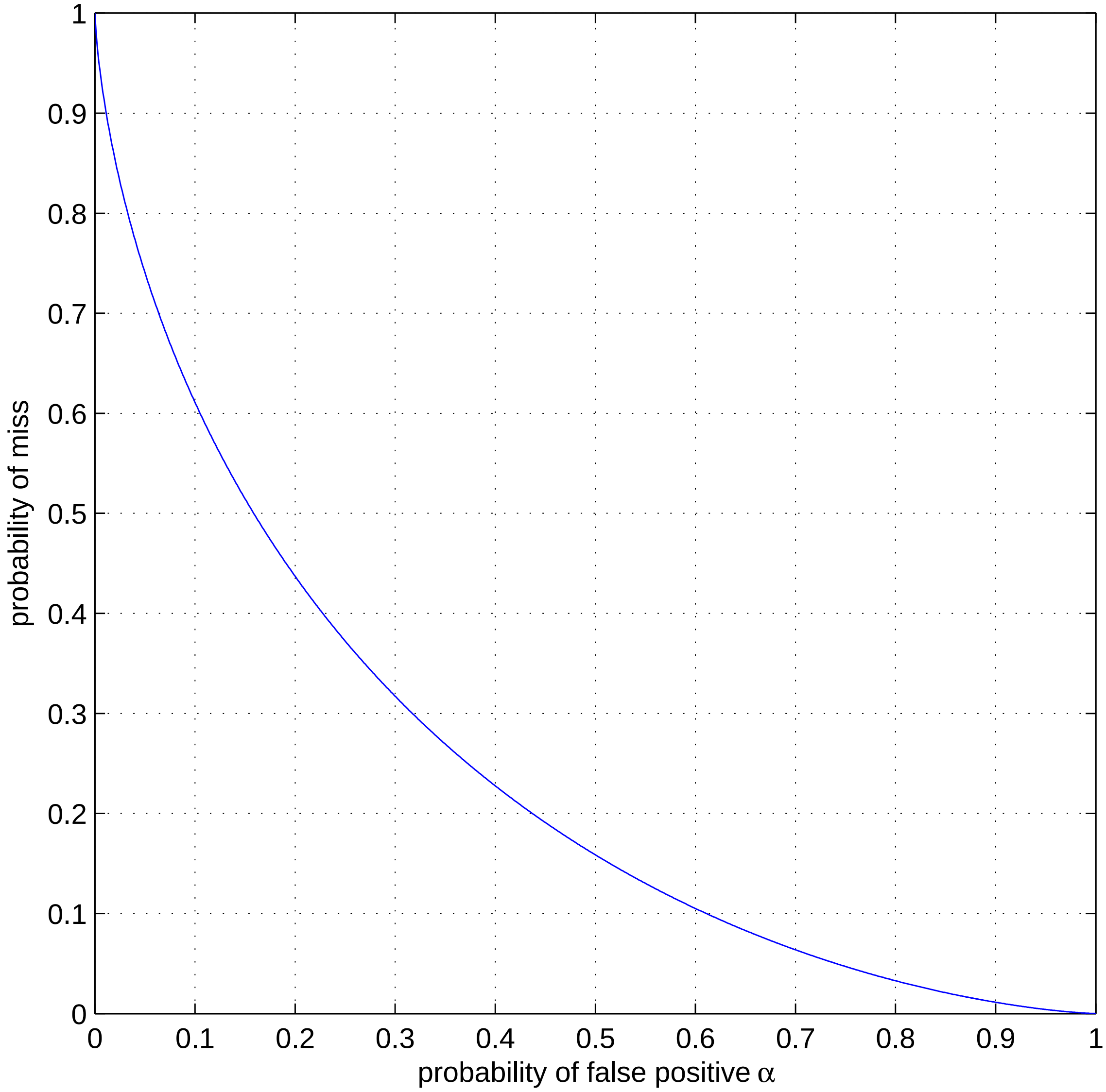

Figure 5: Probability of a miss as a function of the probability of false detection (Kay II:3.1).

4. 3 points. Kay II:3.4

Solution: From Kay equation 3.8, we have the probability of detection

$$\beta = P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{NA^2}{\sigma^2}}\right) \tag{28}$$

where $P_{FA} = \alpha = 10^{-4}$ is the desired probability of false alarm. We want a probability of detection $P_D = 0.99$. We have to do a little algebra to get

$$Q^{-1}(P_D) = Q^{-1}(P_{FA}) - \sqrt{\frac{NA^2}{\sigma^2}} \tag{29}$$

$$\Leftrightarrow \quad \sqrt{\frac{NA^2}{\sigma^2}} = Q^{-1}(10^{-4}) - Q^{-1}(0.99) \tag{30}$$

$$\Leftrightarrow \quad N = \frac{\left(Q^{-1}(10^{-4}) - Q^{-1}(0.99)\right)^2}{0.001} \tag{31}$$

where the denominator in the last equality results from $10\log_{10}(A^2/\sigma^2) = 30\text{dB}$. Plug this into Matlab

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>> N = (qfuncinv(1e-4)-qfuncinv(0.99))^2/0.001
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to get $N = 36546.43$, which means you really need $N = 36547$ samples to get the desired detection of probability and false alarm.