ECE531 Homework Assignment Number 3 Solution

Due by 8:50pm on Wednesday 16-Feb-2011

Make sure your reasoning and work are clear to receive full credit for each problem.

1. 4 points. Specify the N-P decision rule and probability of detection for the following hypothesis testing problem

$$\mathcal{H}_0 : p_Y(y\,;\mathcal{H}_0) = e^{-y}u(y)$$
$$\mathcal{H}_1 : p_Y(y\,;\mathcal{H}_1) = 2e^{-2y}u(y)$$

as a function of the significance level α where $u(y)$ is the usual unit step function equal to one for $y \geq 0$ and equal to zero otherwise.

Solution: All N-P detectors will be of the form

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } L(y) > v \\ \gamma & \text{if } L(y) = v \\ 0 & \text{if } L(y) < v \end{cases} \tag{1}$$

Note that the randomization here is unnecessary since the distributions are continuous. So we can set $\gamma = 0$ for convenience. The likelihood ratio $L(y) = \frac{2e^{-2y}}{e^{-y}} = 2e^{-y}$ for $y \geq 0$. We can write the threshold test

$$\begin{aligned} & L(y) > v \\ \Leftrightarrow\quad & 2e^{-y} > v \\ \Leftrightarrow\quad & y < -\ln(v/2) \\ \Leftrightarrow\quad & y < v' \end{aligned}$$

hence

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } y < v' \\ 0 & \text{if } y \geq v' \end{cases} \tag{2}$$

where v' is selected to satisfy the significance level of the test, i.e. we want to find a value of $v' \geq 0$ such that

$$P_{\mathsf{fp}} = \int_0^{v'} p_Y(y\,;\mathcal{H}_0)\,dy = \int_0^{v'} e^{-y}\,dy = 1 - e^{-v'} = \alpha. \tag{3}$$

From this last equation, we see that $v' = -\ln(1-\alpha)$. Hence, the final explicit form of the N-P decision rule with significance level α is

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } y < -\ln(1-\alpha) \\ 0 & \text{if } y \geq -\ln(1-\alpha) \end{cases} \tag{4}$$

and the probability of detection is

$$P_D = \int_0^{-\ln(1-\alpha)} p_Y(y\,;\mathcal{H}_1)\,dy = \int_0^{-\ln(1-\alpha)} 2e^{-2y}\,dy = 1 - (\alpha-1)^2 = \beta. \tag{5}$$

Figure 1 shows a plot of the significance level versus the probability of detection.

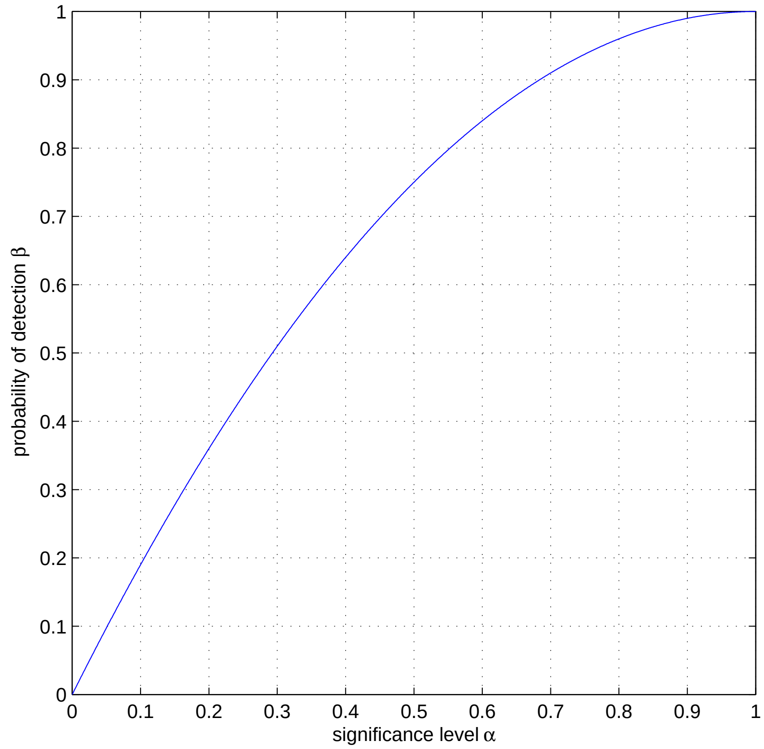

Figure 1: Probability of detection versus significance level for problem 1.

2. 4 points. Specify the N-P decision rule and probability of detection for the following hypothesis testing problem

$$\mathcal{H}_0 : y \sim \mathcal{N}(0,1)$$
$$\mathcal{H}_1 : y \sim \mathcal{U}(-1,1)$$

as a function of the significance level α.

Solution: All N-P detectors will be of the form

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } L(y) > v \\ \gamma & \text{if } L(y) = v \\ 0 & \text{if } L(y) < v \end{cases} \tag{6}$$

Note that the randomization here is unnecessary since the distributions are continuous. So we can set $\gamma = 0$ for convenience. The likelihood ratio

$$L(y) = \begin{cases} \frac{1/2}{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}} & |y| < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

It should be clear at this point that we would never decide $\mathcal{H}_1$ if $|y| \geq 1$ since we would never see an observation in that range if $\mathcal{H}_1$ were true. We still need to figure out what to do when we get an observation in the range $|y| < 1$, however.

Focusing on the case when $|y| < 1$, we can write

$$\begin{aligned} & L(y) > v \\ \Leftrightarrow \quad & \frac{1/2}{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}} > v \\ \Leftrightarrow \quad & y^2/2 > \ln(2v/\sqrt{2\pi}) \\ \Leftrightarrow \quad & |y| > v'. \end{aligned}$$

This should make intuitive sense due to the symmetry of the distributions and the fact that the uniform distribution is more likely (relative to the normal distribution) as we move away from the origin (up to $|y| = 1$). Hence, putting all of our results together, we have

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } v' < |y| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where v' is selected to satisfy the significance level of the test. We find the value of $0 \le v' < 1$ by computing

$$P_{\mathsf{fp}} = \int_{v'}^{1} p_Y(y\,;\mathcal{H}_0)\,dy + \int_{-1}^{-v'} p_Y(y\,;\mathcal{H}_0)\,dy \tag{9}$$

$$= \int_{v'}^{1} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy + \int_{-1}^{-v'} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy \tag{10}$$

$$= 2(\Phi(1) - \Phi(v')) = \alpha. \tag{11}$$

Hence $v' = \Phi^{-1}(\Phi(1) - \alpha/2)$, where $\Phi(x)$ is the usual cdf of a zero-mean unit-variance Guassian random variable. Figure 2 is a plot of v' as a function of α.

Something interesting is going on here. When $\Phi(1) - \alpha/2 = 1/2$, which occurs when $\alpha \approx 0.6827$, the threshold $v' = 0$. This means that we decide $\mathcal{H}_1$ whenever an observation is received between zero and one, otherwise we decide $\mathcal{H}_0$. The probability of *detection* in this case is one (why?). This implies that increasing the significance level α to values greater than 0.6827 can't improve the probability of detection. Hence, we should be careful and say

$$v' = \begin{cases} \Phi^{-1}(\Phi(1) - \alpha/2) & 0 \le \alpha \le 0.6827 \\ 0 & \alpha > 0.6827. \end{cases} \tag{12}$$

The probability of detection is then

$$\begin{aligned} P_D &= \int_{v'}^{1} p_Y(y\,;\mathcal{H}_1)\,dy + \int_{-1}^{-v'} p_Y(y\,;\mathcal{H}_1)\,dy \\ &= \int_{v'}^{1} (1/2)\,dy + \int_{-1}^{-v'} (1/2)\,dy \\ &= 1 - v' = \beta. \end{aligned}$$

with v' defined in (12). This agrees with our earlier comments that the probability of detection is one when the decision threshold $v' = 0$. Finally, Figure 3 shows a plot of the significance level versus the probability of detection. Again, we see the probability of detection is one when $\alpha > 0.6827$.

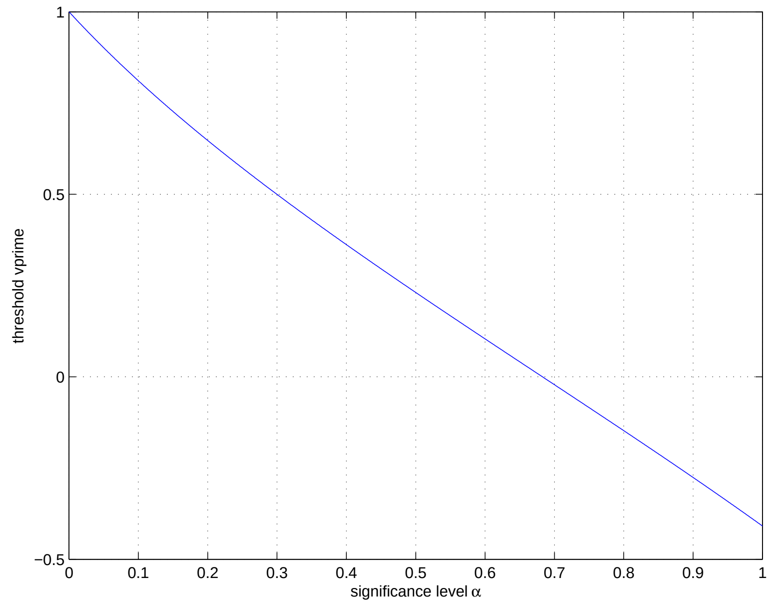

Figure 2: Decision threshold v' versus significance level α.

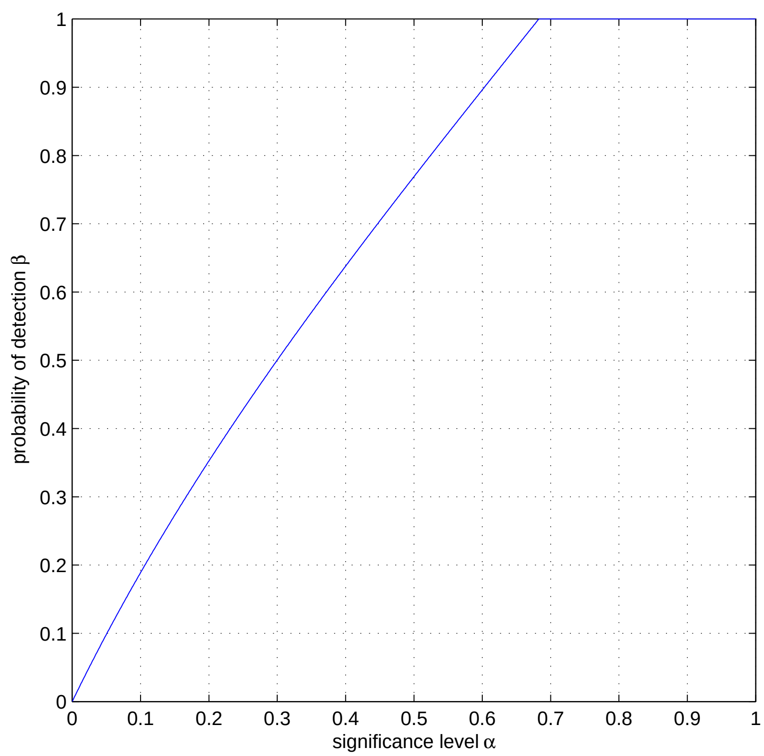

Figure 3: Probability of detection versus significance level for problem 2.

3. 4 points. Specify the Bayes decision rule for problem 1 as a function of the prior probability $\pi_0 = \text{Prob}\{\text{state is } x_0\}$ assuming the uniform cost assignment (UCA).

Solution: Since this is a simple binary hypothesis testing problem, all Bayes decision rules will be of the form

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } L(y) > v \\ 0/1 & \text{if } L(y) = v \\ 0 & \text{if } L(y) < v. \end{cases}$$

with

$$v \quad := \quad \frac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})}.$$

Under the UCA, this simplifies to

$$v \quad := \quad \frac{\pi_0}{1 - \pi_0}$$

since $\pi_1 = 1 - \pi_0$. We already have $L(y)$ from problem 1, so we can write

$$\begin{aligned} & L(y) > v \\ \Leftrightarrow \quad & 2e^{-y} > \frac{\pi_0}{1 - \pi_0} \\ \Leftrightarrow \quad & y < -\ln\left(\frac{\pi_0}{2(1 - \pi_0)}\right) \end{aligned}$$

Hence, our Bayes decision rule is

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } y < -\ln\left(\frac{\pi_0}{2(1-\pi_0)}\right) \\ 0/1 & \text{if } y = -\ln\left(\frac{\pi_0}{2(1-\pi_0)}\right) \\ 0 & \text{if } y > -\ln\left(\frac{\pi_0}{2(1-\pi_0)}\right). \end{cases}$$

Sanity checks: If $\pi_0 \to 0$, the threshold goes to infinity and we always decide $\mathcal{H}_1$. This makes sense because the prior likelihood of $\mathcal{H}_0$ is very small. Similarly, if $\pi_0 \geq 2/3$, the threshold is zero or a negative number and we never decide $\mathcal{H}_1$ (we always decide $\mathcal{H}_0$). This makes sense because the prior likelihood of $\mathcal{H}_0$ is close to one.

The Bayes risk of this decision rule is then

$$r(\delta^{B\pi}, \pi_0) = \begin{cases} \pi_0 R_0(\delta^{B\pi}) + (1 - \pi_0) R_1(\delta^{B\pi}) & 0 \leq \pi_0 \leq 2/3 \\ 1 - \pi_0 & 2/3 < \pi_0 \leq 1 \end{cases} \tag{13}$$

and the conditional risks for $\pi_0 \in [0, 2/3]$ can be computed as

$$\begin{aligned} R_0(\delta^{B\pi}) &= \int_0^{-\ln\left(\frac{\pi_0}{2(1-\pi_0)}\right)} e^{-y}\, dy = \frac{3}{2} - \frac{1}{2(1 - \pi_0)} \\ R_0(\delta^{B\pi}) &= \int_{-\ln\left(\frac{\pi_0}{2(1-\pi_0)}\right)}^{\infty} 2e^{-2y}\, 1y = \frac{\pi_0^2}{4(1 - \pi_0)^2}. \end{aligned}$$

Putting it all together and simplifying a bit, we have

$$r(\delta^{B\pi}, \pi_0) = \begin{cases} \frac{4\pi_0 - 5\pi_0^2}{4(1-\pi_0)} & 0 \leq \pi_0 \leq 2/3 \\ 1 - \pi_0 & 2/3 < \pi_0 \leq 1. \end{cases} \tag{14}$$

Figure 4 plots the Bayes risk as a function of π_0.

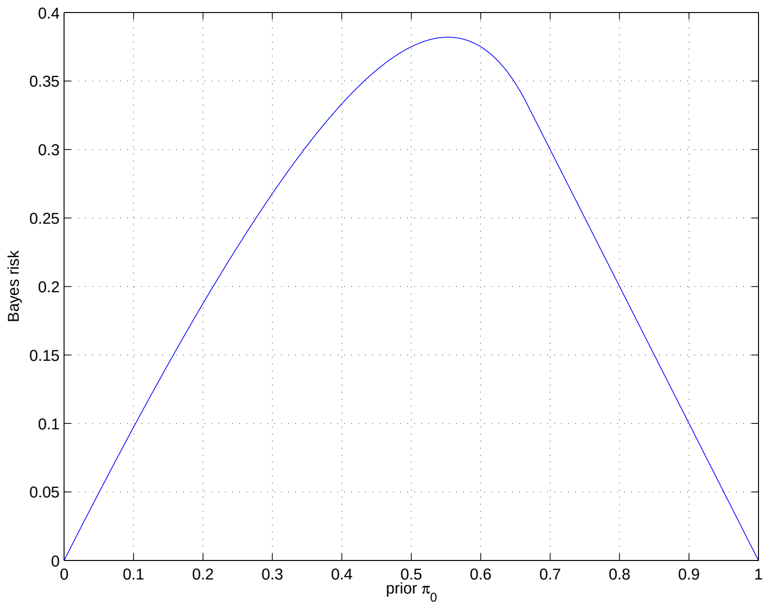

Figure 4: Bayes risk versus prior probability π_0 for problem 3.

Comment: I think it is interesting here that the Bayes detector ignores the observation if the prior $\pi_0 \geq 2/3$. We are trying to distinguish between samples drawn from two different exponential distributions, and the slower decaying distribution seems to have some sort of "dominance" property over the faster decaying distribution. The reason for this is if you compare the functions $f_0(y) = \frac{2}{3}e^{-y}u(y)$ and $f_1(y) = \frac{1}{3}2e^{-2y}u(y)$, you will see that $f_0(y) \geq f_1(y)$ for all y. There is no value of $\pi_0 > 0$ for which $f_1(y) \geq f_0(y)$ for all y, hence we would never resort to the decision rule "always decide $\mathcal{H}_1$" unless $\pi_0 = 0$.

4. 5 points. (Revisiting the quality checking problem from HW2): Suppose you work in a microprocessor manufacturing facility and that, before boxing and shipping, each microprocessor undergoes a quality check to avoid shipping defective units. The quality checking machine has the following characteristics:

   - It declares good microprocessors to be defective (D) with probability $p = 0.15$.
   - It declares defective microprocessors to be good (G) with probability $q = 0.03$.

   Suppose there are n such quality checking machines that give independent results with the same probabilities. Also let $\mathcal{H}_0$ be the hypothesis that the microprocessor is good and let $\mathcal{H}_1$ be the hypothesis that the microprocessor is defective. You receive some new information from manufacturing and customer service saying that

   - It costs \$200 to replace a defective microprocessor after it has been shipped to the customer.
   - It costs \$50 each time good microprocessors are discarded.
   - It costs nothing when good microprocessors are shipped and bad microprocessors are discarded.

   Assume $n = 2$. Determine the Bayes decision rule as a function of the prior probability $\pi_0 = \text{Prob}\{\text{microprocessor is good}\}$. Comment on your decision rule in the extreme cases when $\pi_0 \to 0$ and $\pi_0 \to 1$. Plot the risk of the Bayes detector as a function of π_0.

   Solution: Since this is a simple binary hypothesis testing problem, all Bayes decision rules will be of the form

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } L(y) > v \\ 0/1 & \text{if } L(y) = v \\ 0 & \text{if } L(y) < v. \end{cases}$$

   with

$$v \quad := \quad \frac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})}.$$

   The costs are $C_{00} = C_{11} = 0$, $C_{01} = 200$ (the cost of deciding the processor is good when it is actually defective) and $C_{10} = 50$ (the cost of deciding the processor is defective when it is actually good). Hence,

$$v \quad := \quad \frac{\pi_0}{4(1 - \pi_0)}$$

   since $\pi_1 = 1 - \pi_0$.

   From HW2, we use the number of D's that we observe as our observation, hence $y \in \{0, 1, 2\}$. We have the conditional probability matrix P and the likelihood vector $L(y)$ also from HW2 as

$$P = \begin{bmatrix} 0.723 & 0.0009 \\ 0.255 & 0.0582 \\ 0.022 & 0.9409 \end{bmatrix} \qquad L = \begin{bmatrix} 0.0012 \\ 0.2282 \\ 42.7682 \end{bmatrix} \tag{15}$$

   Note that $\frac{\pi_0}{4(1-\pi_0)} > x$ is the same thing as saying $\pi_0 > \frac{4x}{1+4x}$. The Bayes decision rule and associated risks are broken down into 4 cases below:

   - when $0 \leq \pi_0 \leq \frac{4\times 0.0012}{1+4\times 0.0012} = 0.0048$, the likelihood ratio $L(y)$ is always larger than v, irrespective of what the machines tell us. Hence, we always decide $\mathcal{H}_1$. This means that, if our prior probability of manufacturing a good microprocessor (let's call this the *yield*) is less than 0.48%, we shouldn't ship anything. The Bayes risk in this case is then

$$r(\delta^{B\pi}, \pi_0) = \pi_0 R_0(\delta^{B\pi}) + (1 - \pi_0) R_1(\delta^{B\pi}) = 50\pi_0. \tag{16}$$

- When $0.0048 = \frac{4\times 0.0012}{1+4\times 0.0012} < \pi_0 \leq \frac{4\times 0.2282}{1+4\times 0.2282} = 0.4772$, the likelihood ratio $L(y)$ is only larger than v when we observe one or two D's. Hence, we decide $\mathcal{H}_1$ if we see one or two D's, otherwise we decide $\mathcal{H}_0$. This means that, if our yield is roughly between 0.5% and 48%, we should ship only microprocessors that pass both quality checks (no D's). The decision matrix in this case is

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{17}$$

  Given the P matrix above and a C matrix

$$C = \begin{bmatrix} 0 & 200 \\ 50 & 0 \end{bmatrix} \tag{18}$$

  we can easily compute the conditional risk vector $R(D) = [13.8750, 0.1800]^\top$ and the associated Bayes risk

$$r(\delta^{B\pi}, \pi_0) = 13.8750\pi_0 + 0.1800(1 - \pi_0) \tag{19}$$

- When $0.4772 = \frac{4\times 0.2282}{1+4\times 0.2282} < \pi_0 \leq \frac{4\times 42.7682}{1+4\times 42.7682} = 0.9942$, the likelihood ratio $L(y)$ is only larger than v when we observe two D's. Hence, we decide $\mathcal{H}_1$ if we see two D's, otherwise we decide $\mathcal{H}_0$. This means that, if our yield is roughly between 48% and 99.4%, we should ship all microprocessors that pass one or both quality checks. The decision matrix in this case is

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

  Given the P and Cmatrices above we can easily compute the conditional risk vector $R(D) = [1.1250, 11.8200]^\top$ and the associated Bayes risk

$$r(\delta^{B\pi}, \pi_0) = 1.1250\pi_0 + 11.8200(1 - \pi_0) \tag{21}$$

- when $0.9942 = \frac{4\times 42.7682}{1+4\times 42.7682} < \pi_0 \leq 1$, the likelihood ratio $L(y)$ is never larger than v, and we always decide $\mathcal{H}_0$. This means that if the manufacturing is good enough so that about 99.42% or better of the microprocessors are good, we should just ship everything. The Bayes risk in this case is then

$$r(\delta^{B\pi}, \pi_0) = \pi_0 R_0(\delta^{B\pi}) + (1 - \pi_0) R_1(\delta^{B\pi}) = 200(1 - \pi_0). \tag{22}$$

So, in the extreme case when $\pi_0 \to 0$, we should ship nothing since it is very unlikely that anything we manufactured is good. In the extreme case when $\pi_0 \to 1$, we should ship everything and not waste time doing quality checks.

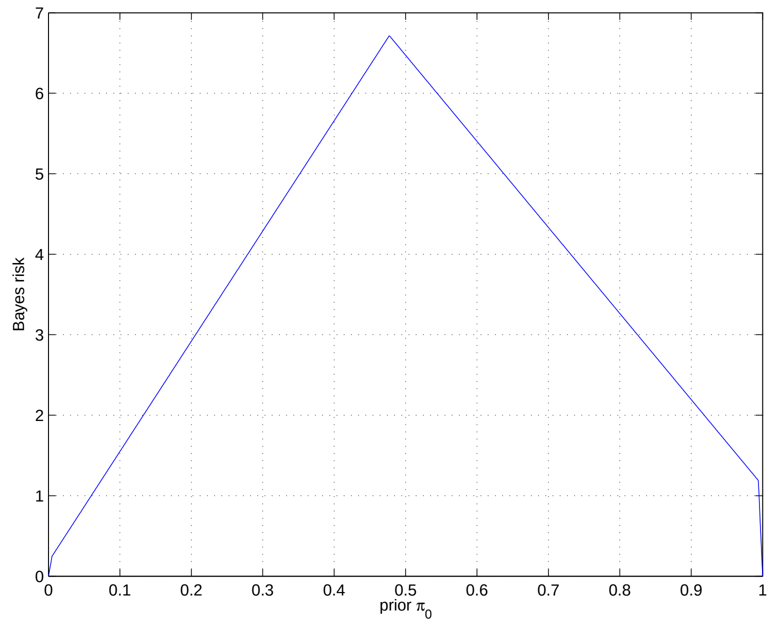

Figure 5: Bayes risk versus prior probability π_0 for problem 4.

5. 4 points. Kay II: 3.18.

Solution: Since this is a simple binary hypothesis testing problem, all Bayes decision rules will be of the form

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } L(y) > v \\ 0/1 & \text{if } L(y) = v \\ 0 & \text{if } L(y) < v. \end{cases}$$

with

$$v \quad := \quad \frac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})}.$$

The MAP decision rule specifies the UCA, so $v = \frac{\pi_0}{1-\pi_0}$. The likelihood ratio in this case is

$$L(y) = \frac{\frac{1}{\sqrt{2\pi \cdot 2}} \exp\left(-\frac{1}{2 \cdot 2} y^2\right)}{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y^2\right)} = \frac{1}{\sqrt{2}} \exp\left(\frac{1}{4} y^2\right). \tag{23}$$

After some algebra, comparison $L(y) > v$ can be equivalently expressed as

$$|y| > 2\sqrt{\ln(\sqrt{2}v))}. \tag{24}$$

When $\pi_0 = 1/2$, we have $v = 1$ and

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } |y| > 2\sqrt{\ln(\sqrt{2}))} \\ 0/1 & \text{if } |y| = 2\sqrt{\ln(\sqrt{2}))} \\ 0 & \text{if } |y| < 2\sqrt{\ln(\sqrt{2}))}. \end{cases}$$

where $\sqrt{\ln(\sqrt{2})} \approx 1.1774$. When $\pi_0 = 3/4$, we have $v = 3$ and

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } |y| > 2\sqrt{\ln(3\sqrt{2})} \\ 0/1 & \text{if } |y| = 2\sqrt{\ln(3\sqrt{2})} \\ 0 & \text{if } |y| < 2\sqrt{\ln(3\sqrt{2})}. \end{cases}$$

where $\sqrt{\ln(3\sqrt{2})} \approx 2.4043$. The decision regions are sketched for both cases in Figure 6 below.

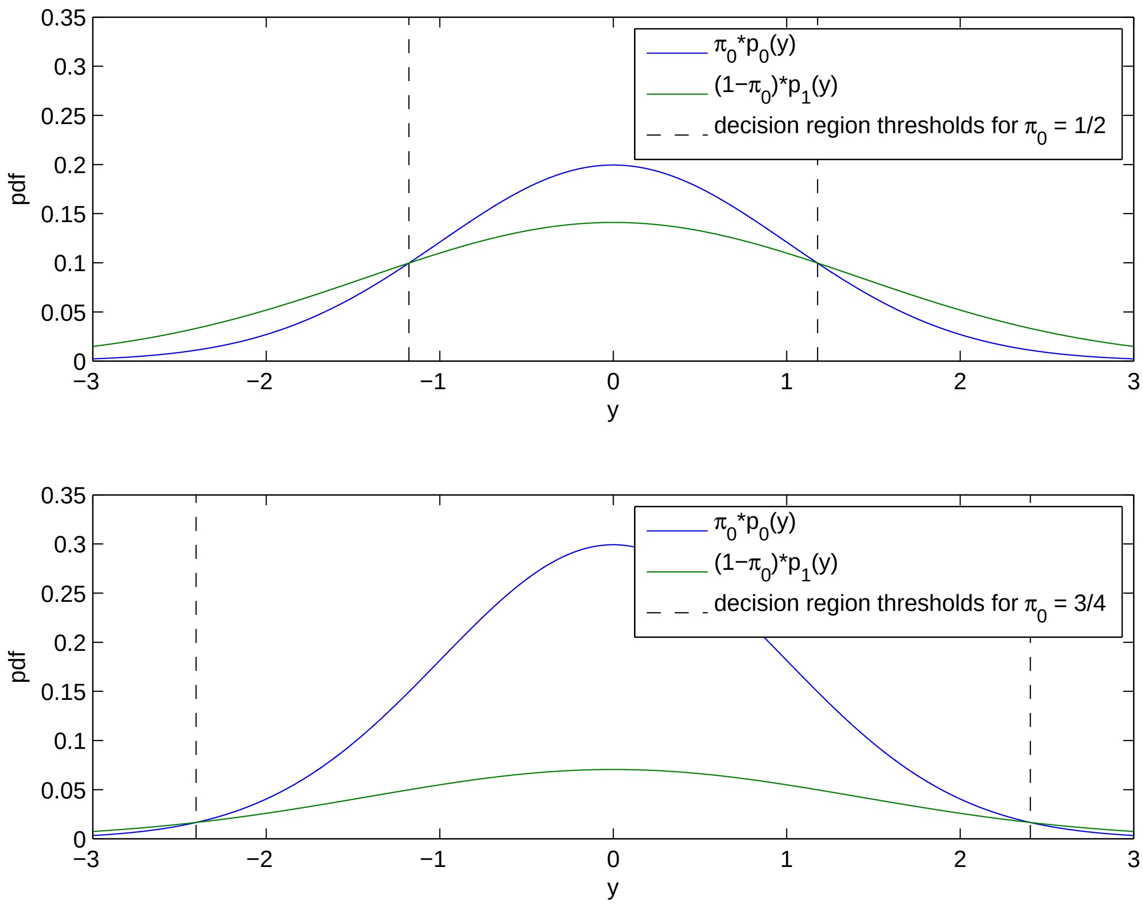

Figure 6: Decision regions for problem 5.

6. 4 points. Kay II: 3.21

Solution: This is *not* a binary hypothesis testing problem, so we need to approach this differently than the previous problems. In this case, we have $N = 3$ and we can write the "commodity" functions

$$\begin{aligned} g_0(y,\pi) &= \pi_0 C_{00} p_0(y) + \pi_1 C_{01} p_1(y) + \pi_2 C_{02} p_2(y) \\ g_1(y,\pi) &= \pi_0 C_{10} p_0(y) + \pi_1 C_{11} p_1(y) + \pi_2 C_{12} p_2(y) \\ g_2(y,\pi) &= \pi_0 C_{20} p_0(y) + \pi_1 C_{21} p_1(y) + \pi_2 C_{22} p_2(y) \end{aligned}$$

The problem specifies equal prior probabilities, so $\pi_0 = \pi_1 = \pi_2 = 1/3$. The goal is to find the minimum probability of error detector, so this means we should use the UCA. So the commodity functions can be simplified to

$$\begin{aligned} g_0(y,\pi) &= p_1(y) + p_2(y) \\ g_1(y,\pi) &= p_0(y) + p_2(y) \\ g_2(y,\pi) &= p_0(y) + p_1(y) \end{aligned}$$

where we've discarded the common 1/3 factor because it doesn't affect the minimization. Note that

$$\begin{aligned} p_0(y) &= \frac{1}{2}\exp(-|y+1|) \\ p_1(y) &= \frac{1}{2}\exp(-|y|) \\ p_2(y) &= \frac{1}{2}\exp(-|y-1|) \end{aligned}$$

so we can write the commodity functions explicitly as

$$\begin{aligned} g_0(y,\pi) &= \exp(-|y|) + \exp(-|y-1|) \\ g_1(y,\pi) &= \exp(-|y+1|) + \exp(-|y-1|) \\ g_2(y,\pi) &= \exp(-|y+1|) + \exp(-|y|) \end{aligned}$$

again discarding common factors since they don't affect the minimization.

Note that $g_0(y,\pi)$ is the minimum cost commodity if two things are true:

$$\begin{aligned} g_0(y,\pi) < g_1(y,\pi) \quad &\Leftrightarrow \quad \exp(-|y|) < \exp(-|y+1|) \quad \Leftrightarrow \quad |y| > |y+1| \\ g_0(y,\pi) < g_2(y,\pi) \quad &\Leftrightarrow \quad \exp(-|y-1|) < \exp(-|y+1|) \quad \Leftrightarrow \quad |y-1| > |y+1| \end{aligned}$$

The range of y over which both of these conditions hold is $y \in \mathcal{Y}_0 = (-\infty, 1/2)$ (you can check this graphically). Hence, we select $\mathcal{H}_0$ if $y \in \mathcal{Y}_0$.

Similarly, $g_1(y,\pi)$ is the minimum cost commodity if two things are true:

$$\begin{aligned} g_1(y,\pi) < g_0(y,\pi) \quad &\Leftrightarrow \quad \exp(-|y+1|) < \exp(-|y|) \quad \Leftrightarrow \quad |y+1| > |y| \\ g_1(y,\pi) < g_2(y,\pi) \quad &\Leftrightarrow \quad \exp(-|y-1|) < \exp(-|y|) \quad \Leftrightarrow \quad |y-1| > |y| \end{aligned}$$

The range of y over which both of these conditions hold is $y \in \mathcal{Y}_1 = (-1/2, 1/2)$ (you can check this graphically). Hence, we select $\mathcal{H}_1$ if $y \in \mathcal{Y}_1$.

Finally, $g_2(y,\pi)$ is the minimum cost commodity if two things are true:

$$\begin{aligned} g_2(y,\pi) < g_0(y,\pi) \quad &\Leftrightarrow \quad \exp(-|y+1|) < \exp(-|y-1|) \quad \Leftrightarrow \quad |y+1| > |y-1| \\ g_2(y,\pi) < g_1(y,\pi) \quad &\Leftrightarrow \quad \exp(-|y|) < \exp(-|y-1|) \quad \Leftrightarrow \quad |y| > |y-1| \end{aligned}$$

The range of y over which both of these conditions hold is $y \in \mathcal{Y}_2 = (1/2, \infty)$ (you can check this graphically). Hence, we select $\mathcal{H}_2$ if $y \in \mathcal{Y}_2$.

With the decision regions all worked out, the last thing to do is compute the probability of error. The probability of making an error conditioned on the initial state being x_0 is

$$\text{Prob(decide } \mathcal{H}_1 \text{ or } \mathcal{H}_2 \,|\, x_0) = \int_{-1/2}^{\infty} \frac{1}{2}\exp(-|y+1|)\,dy = \frac{1}{2\sqrt{e}} \tag{25}$$

By symmetry, we can also say the probability of making an error conditioned on the initial state being x_2 is

$$\text{Prob}(\text{decide } \mathcal{H}_0 \text{ or } \mathcal{H}_1 \,|\, x_2) = \int_{-\infty}^{1/2} \frac{1}{2} \exp(-|y-1|)\, dy = \frac{1}{2\sqrt{e}} \tag{26}$$

And finally, the probability of making an error conditioned on the initial state being x_1 is

$$\text{Prob}(\text{decide } \mathcal{H}_0 \text{ or } \mathcal{H}_2 \,|\, x_1) = 1 - \int_{-1/2}^{1/2} \frac{1}{2} \exp(-|y|)\, dy = \frac{1}{\sqrt{e}} \tag{27}$$

Hence the total probability of error is

$$\begin{aligned} P_e &= \frac{1}{3}\text{Prob}(\text{decide } \mathcal{H}_1 \text{ or } \mathcal{H}_2 \,|\, x_0) + \frac{1}{3}\text{Prob}(\text{decide } \mathcal{H}_0 \text{ or } \mathcal{H}_2 \,|\, x_1) + \frac{1}{3}\text{Prob}(\text{decide } \mathcal{H}_0 \text{ or } \mathcal{H}_2 \,|\, x_1) \\ &= \frac{2}{3\sqrt{e}} \approx 0.4044 \end{aligned}$$