ECE531 Homework Assignment Number 4 Solution

Due by 8:50pm on Wednesday 23-Feb-2011

Make sure your reasoning and work are clear to receive full credit for each problem.

1. 5 points total. Suppose you have a communication system in which two signals are transmitted to convey one bit of information. The signals are

$$x_0 : \cos\left(\frac{\pi}{4}n\right)$$
$$x_1 : \cos\left(\frac{\pi}{4}n + \phi\right)$$

for $n = 0, 1, \dots, 3$ where $-\pi < \phi \leq \pi$ is known. The observation at the detector is corrupted by stationary zero-mean additive Gaussian noise $w[n]$ with covariance

$$\mathrm{E}[w[n]w[m]] = \begin{cases} 0.50 & n = m \\ 0.25 & |n-m| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) 3 points. Suppose $\phi = \pi/2$. Determine the minimum probability of error decision rule and its probability of error as a function of the prior probability π_0.

Solution: The signals in this case are $s_0 = [1, 0.7071, 0, -0.7071]^\top$ and $s_1 = [0, -0.7071, -1, -0.7071]^\top$. The noise covariance is

$$\Sigma = \begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}$$

and Matlab tells us that the inverse covariance matrix is

$$\Sigma^{-1} = \begin{bmatrix} 3.2 & -2.4 & 1.6 & -0.8 \\ -2.4 & 4.8 & -3.2 & 1.6 \\ 1.6 & -3.2 & 4.8 & -2.4 \\ -0.8 & 1.6 & -2.4 & 3.2 \end{bmatrix}.$$

We can use Matlab to compute the Cholesky factorization of this inverse covariance matrix such that $\Sigma^{-1} = S^\top S$ as

$$S = \begin{bmatrix} 1.7889 & -1.3416 & 0.8944 & -0.4472 \\ 0 & 1.7321 & -1.1547 & 0.5774 \\ 0 & 0 & 1.6330 & -0.8165 \\ 0 & 0 & 0 & 1.4142 \end{bmatrix}.$$

You can confirm that $S\Sigma S^\top = I$, hence this S matrix decorrelates the noise. The coordinate transformed signal vectors are then

$$\bar{s}_0 = Ss_0 = \begin{bmatrix} 1.1564 \\ 0.8165 \\ 0.5774 \\ -1.0000 \end{bmatrix} \qquad \bar{s}_1 = Ss_1 = \begin{bmatrix} 0.3705 \\ -0.4783 \\ -1.0556 \\ -1.0000 \end{bmatrix}$$

and the coordinate transformed difference vector is $\bar{s} = [-0.7859, -1.2948, -1.6330, 0]^\top$. From the week 4 slides, we know our decision variable is $Z = \bar{s}^\top Y$. Conditioned on $x = s_j$, we know that Z is Gaussian distributed with mean and variance

$$\mu_j = \bar{s}^\top \bar{s}_j$$
$$\sigma^2 = \bar{s}^\top \bar{s}$$

Hence $\mu_0 = -2.9088$, $\mu_1 = 2.0520$, and $\sigma^2 = 4.9608$. We also know that the threshold for the minimum probability of error detector of a simple binary hypothesis test with $\mathcal{H}_0 : y \sim \mathcal{N}(\mu_0, \sigma^2)$ and $\mathcal{H}_1 : y \sim \mathcal{N}(\mu_1, \sigma^2)$ is

$$\tau = \frac{\mu_0 + \mu_1}{2} + \frac{\sigma^2}{\mu_1 - \mu_0} \ln \frac{\pi_0}{1 - \pi_0} = -0.4284 + \ln \frac{\pi_0}{1 - \pi_0}$$

hence the minimum probability of error decision rule is then

$$\rho(\bar{y}) = \begin{cases} 1 & \bar{s}^\top \bar{y} \geq -0.4284 + \ln \frac{\pi_0}{1-\pi_0} \\ 0 & \qquad < -0.4284 + \ln \frac{\pi_0}{1-\pi_0} \end{cases}$$

The actual error probability is

$$\begin{aligned} P_e &= \pi_0 \Pr(Z \geq \tau \,|\, x = s_0) + (1 - \pi_0) \Pr(Z < \tau \,|\, x = s_1) \\ &= \pi_0 Q\left(\frac{\tau - \mu_0}{\sigma}\right) + (1 - \pi_0) Q\left(\frac{\mu_1 - \tau}{\sigma}\right) \\ &= \pi_0 Q\left(\frac{2.4804 + \ln \frac{\pi_0}{1-\pi_0}}{2.2273}\right) + (1 - \pi_0) Q\left(\frac{2.4804 - \ln \frac{\pi_0}{1-\pi_0}}{2.2273}\right) \end{aligned}$$

which is plotted in Figure 1. Note this agrees with our result when the prior $\pi_0 = 0.5$ in that we know $P_e = Q(||\bar{s}||/2) = 0.1327$ in this case.

(b) 2 points. Suppose $\pi_0 = 0.5$ and you are allowed to design the signal in state 1 to minimize the probability of error. Determine the value of ϕ that minimizes the probability of error.

Solution: Note that s_0 is fixed here. By varying ϕ, we are only changing s_1. We want to find the value of ϕ that maximizes $||\bar{s}||$ since that will lead to the minimum probability of error. You can do some analysis here to find the value of ϕ that maximizes $||\bar{s}||$, but you will end up with a problem $f(\phi) = 0$ that can only be solved numerically (since $f(\phi)$ is composed of trigonometric functions). Instead, we can use Matlab to find the best value of ϕ as follows:

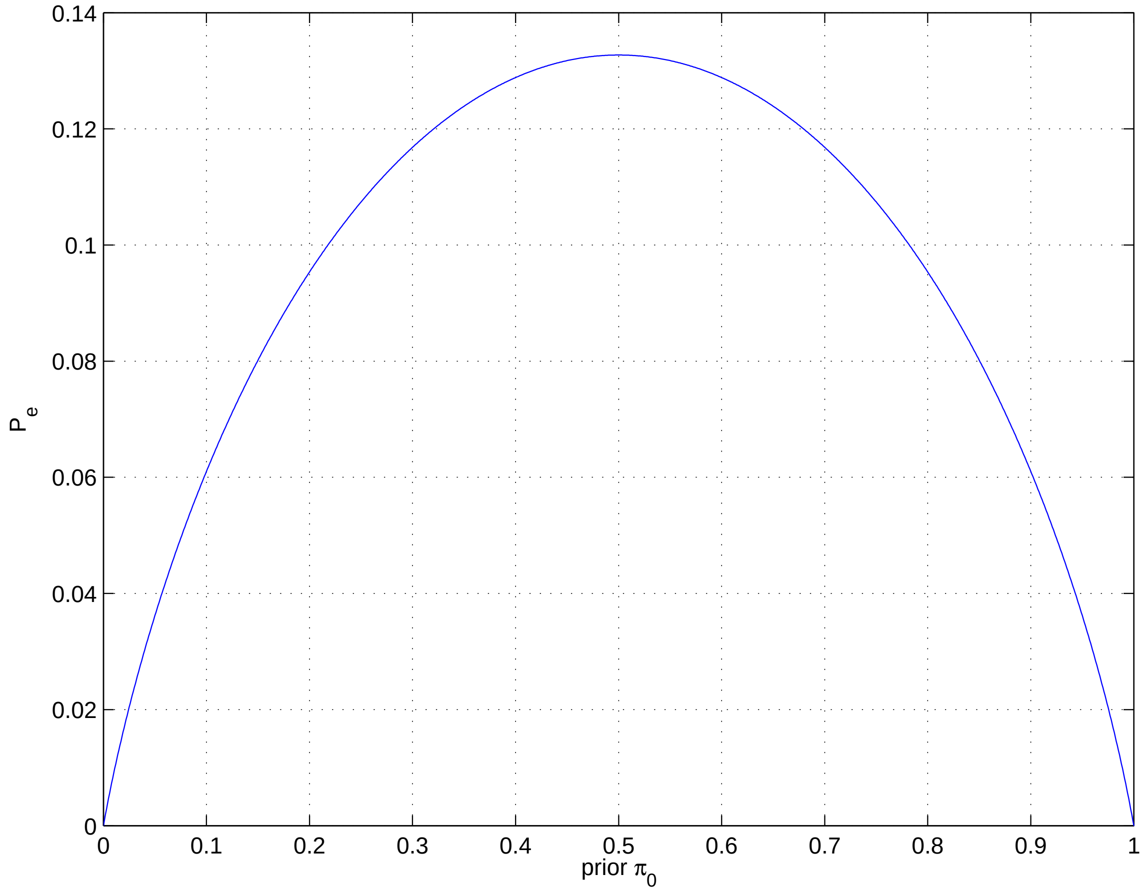

Figure 1: Probability of error versus prior probability level for problem 1a.

```
% ECE531 HW4 P1b

phitest = -pi:0.001:pi;            % range of phi

% noise covariance matrix
SIGMA = [0.5 0.25 0 0 ; 0.25 0.5 0.25 0; 0 0.25 .5 0.25; 0 0 0.25 0.5];
SIGMAINV = inv(SIGMA);
S = chol(SIGMAINV);

% generate s0 signal and transform coordinates
n = [0:3]';
s0 = cos(pi/4*n);
s0bar = S*s0;

% allocate space for the norm of sbar
sbarnorm = zeros(1,length(phitest));

% find value of phi that maximizes the norm of sbar
i1 = 0;
for phi=phitest,
    i1 = i1+1;
    s1 = cos(pi/4*n+phi);
    s1bar = S*s1;
    sbar = s1bar-s0bar;
    sbarnorm(i1) = sqrt(sbar'*sbar);
end
[junk,index] = max(sbarnorm);

% make plot
plot(phitest,sbarnorm,phitest(index),sbarnorm(index),'rp');
xlabel('angle \phi');
ylabel('||sbar||');
grid on
```

A plot of $||\bar{s}||$ as a function of ϕ is shown in Figure 2. The optimum value for ϕ is $\phi \approx -2.9446$, which leads to a $P_e \approx Q(||\bar{s}||/2) \approx 0.0330$. This is an improvement of about a factor of four with respect to part (a).

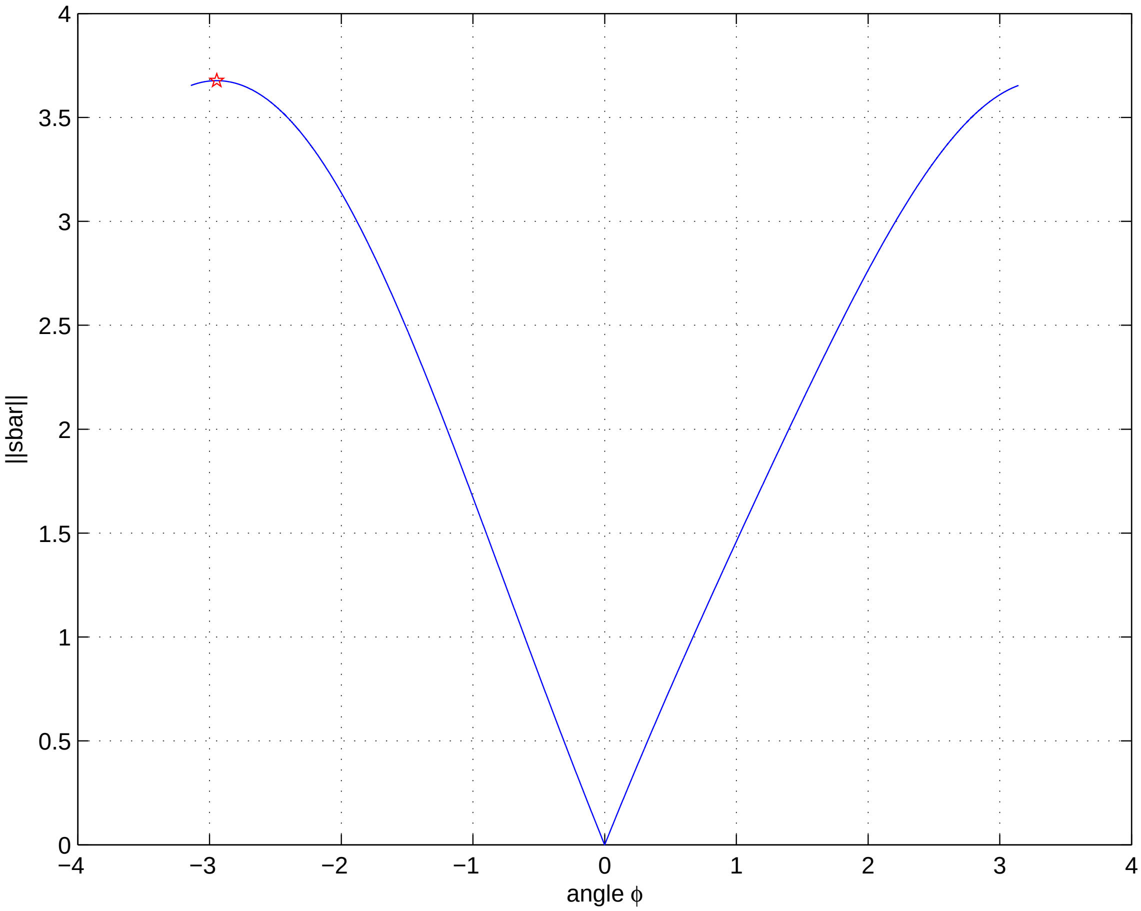

Figure 2: $||\bar{s}||$ versus ϕ for problem 1b.

2. 4 points. Kay II: 4.6

Solution: This is a simple binary hypothesis testing problem. Since the noise is white in this problem, there is no need for decorrelation. It is convenient to scale the observation so that the noise has unit variance. Hence, let's let the scaled (vector) observation $\bar{y} = \frac{1}{\sigma}y$. We also define the scaled signal vectors $\bar{s}_0 = \frac{1}{\sigma}s_0 = [0,\dots,0]^\top$ and $\bar{s}_1 = \frac{1}{\sigma}s_1 = [A/\sigma, Ar/\sigma, Ar^2/\sigma, \dots, Ar^{N-1}/\sigma]^\top$. Defining $\bar{s} = \bar{s}_1 - \bar{s}_0$, the N-P detector will be of the form

$$\rho(\bar{y}) = \begin{cases} 1 & \bar{s}^\top\bar{y} \geq \ln v + \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2) \\ 0 & \qquad < \ln v + \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2) \end{cases}$$

where we select v to achieve the desired false positive probability. Let's let $v' = \ln v + \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2)$. The decision statistic in this problem is $Z = \bar{s}^\top\bar{Y}$. Conditioned on $x = s_j$, we know that Z is Gaussian distributed with mean and variance

$$\mu_j = \bar{s}^\top\bar{s}_j$$
$$\sigma^2 = \bar{s}^\top\bar{s}$$

Hence $\mu_0 = 0$, $\mu_1 = \frac{A^2}{\sigma^2}\sum_{k=0}^{N-1} r^{2k}$, and $\sigma^2 = \frac{A^2}{\sigma^2}\sum_{k=0}^{N-1} r^{2k}$. Given a threshold v', the probability of false positive is then

$$P_{\mathsf{fp}} = Q\left(\frac{v'}{\frac{A}{\sigma}\sqrt{\sum_{k=0}^{N-1} r^{2k}}}\right) = \alpha$$

Hence

$$v' = \frac{AQ^{-1}(\alpha)}{\sigma}\sqrt{\sum_{k=0}^{N-1} r^{2k}}$$

The probability of detection is then

$$\begin{aligned} P_D &= Q\left(\frac{v' - \mu_1}{\sigma}\right) \\ &= Q\left(\frac{\frac{AQ^{-1}(\alpha)}{\sigma}\sqrt{\sum_{k=0}^{N-1} r^{2k}} - \frac{A^2}{\sigma^2}\sum_{k=0}^{N-1} r^{2k}}{\frac{A}{\sigma}\sqrt{\sum_{k=0}^{N-1} r^{2k}}}\right) \\ &= Q\left(Q^{-1}(\alpha) - \frac{A}{\sigma}\sqrt{\sum_{k=0}^{N-1} r^{2k}}\right). \end{aligned}$$

- When $0 < r < 1$ and $N \to \infty$, the term $\sum_{k=0}^{N-1} r^{2k} \to \frac{1}{1-r^2}$. In this case,

  $$P_D = Q\left(Q^{-1}(\alpha) - \frac{A}{\sigma}\frac{1}{\sqrt{1-r^2}}\right).$$

  Clearly, P_D improves as $r \to 1$. This is because the total "energy" in the signal s_1 gets larger as $r \to 1$ while the noise variance stays constant.
- When $r \geq 1$ and $N \to \infty$, the term $\sum_{k=0}^{N-1} r^{2k} \to \infty$. In this case,

  $$P_D = Q\left(Q^{-1}(\alpha) - \infty\right) = Q(-\infty) = 1.$$

  The probability of detection in this case is one because the signal s_1 has infinite energy.

3. 4 points. Kay II: 4.7

Solution: We can follow the same approach as the previous problem. Let the scaled (vector) observation $\bar{y} = \frac{1}{\sigma}y$. We also define the scaled signal vectors $\bar{s}_0 = \frac{1}{\sigma}s_0 = [0,\ldots,0]^\top$ and $\bar{s}_1 = \frac{1}{\sigma}s_1 = [A/\sigma, A\cos(\pi/2)/\sigma, A\cos(\pi)/\sigma, \ldots, A\cos(12\pi)/\sigma]^\top$. Defining $\bar{s} = \bar{s}_1 - \bar{s}_0$, the N-P detector will be of the form

$$\rho(\bar{y}) \quad = \quad \begin{cases} 1 & \bar{s}^\top \bar{y} \quad \geq \ln v + \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2) \\ 0 & \qquad\ < \ln v + \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2) \end{cases}$$

where we select v to achieve the desired false positive probability. Let's let $v' = \ln v + \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2)$. The decision statistic in this problem is $Z = \bar{s}^\top \bar{Y}$. Conditioned on $x = s_j$, we know that Z is Gaussian distributed with mean and variance

$$\mu_j = \bar{s}^\top \bar{s}_j$$
$$\sigma^2 = \bar{s}^\top \bar{s}$$

Hence $\mu_0 = 0$, $\mu_1 = 13A^2$, and $\sigma^2 = 13A^2$. Given a threshold v', the probability of false positive is then

$$P_{\mathsf{fp}} = Q\left(\frac{v'}{|A|\sqrt{13}}\right) = \alpha = 10^{-8}$$

Hence

$$v' = Q^{-1}(10^{-8})|A|\sqrt{13} \approx 20.2344|A|.$$

The probability of detection is then

$$\begin{aligned} P_D &= Q\left(\frac{v' - \mu_1}{\sigma}\right) \\ &= Q\left(\frac{20.2344|A| - 13A^2}{|A|\sqrt{13}}\right) \\ &= Q\left(5.6120 - |A|\sqrt{13}\right). \end{aligned}$$

As expected, the probability of detection is clearly increasing as A increases. The sign of A does not affect the result.

4. 4 points. Suppose the state $x \sim \mathcal{U}(0,2)$ and we receive one observation $Y = x + W$ where $p_W(t) = e^{-t}u(t)$. Determine the Bayes decision rule to decide between the composite hypotheses

$$\mathcal{H}_0 : 0 \leq x < 1$$
$$\mathcal{H}_1 : 1 \leq x \leq 2.$$

Solution: This is a binary composite hypothesis testing problem with both hypotheses have an infinite number of states associated with them. The general approach to this sort of problem is to compute the commodity costs and create a decision rule that selects the minimum commodity cost. So, in preparation for computing these commodity costs, we need to state the following facts from the problem description:

$$\begin{aligned}
\mathcal{X} &= [0,2] \\
\pi(x) &= 1/2 \text{ for } x \in \mathcal{X} \\
C_0(x) &= \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x \leq 2 \end{cases} \\
C_1(x) &= \begin{cases} 1 & 0 \leq x < 1 \\ 0 & 1 \leq x \leq 2 \end{cases} \\
p_x(y) &= e^{-(y-x)}u(y-x) \text{ for } x \in \mathcal{X}
\end{aligned}$$

where the last equality comes from the fact that $Y = x+W$, hence x just shifts the distribution of W to the right. The commodity costs are then

$$\begin{aligned}
g_0(y,\pi) &= \frac{1}{2}\int_1^2 e^{-(y-x)}u(y-x)\,dx \\
g_1(y,\pi) &= \frac{1}{2}\int_0^1 e^{-(y-x)}u(y-x)\,dx
\end{aligned}$$

We can factor out the common (positive valued) terms since they don't affect selecting the minimum. So, equivalently, the commodity costs can be expressed as

$$\begin{aligned}
g_0(y,\pi) &= \int_1^2 e^{x}u(y-x)\,dx \\
g_1(y,\pi) &= \int_0^1 e^{x}u(y-x)\,dx
\end{aligned}$$

Let's look at these commodity costs in three cases:

- Case 1: $0 < y \leq 1$. Note that $u(y-x)$ is zero for all $x > y$. Hence, when we get an observation in this region, $u(y-x)$ is certainly zero for all $x > 1$. Hence $g_0(y,\pi) = 0$ and

$$g_1(y,\pi) = \int_0^y e^{x}\,dx = e^{y} - 1 > 0.$$

  Hence, we decide $\mathcal{H}_0$ when we get observations in this region.
- Case 2: $1 < y \leq 2$. When we get an observation in this region, the commodity costs are

$$\begin{aligned}
g_0(y,\pi) &= \int_1^y e^{x}\,dx = e^{y} - e \\
g_1(y,\pi) &= \int_0^1 e^{x}\,dx = e - 1
\end{aligned}$$

and a little bit of algebra reveals that $g_1(y,\pi) < g_0(y,\pi)$ if and only if $y > \ln(2e-1) \approx 1.4899$. So we decide $\mathcal{H}_0$ if $y \leq \ln(2e-1)$ and we decide $\mathcal{H}_1$ if $y > \ln(2e-1)$.

- Case 3: $y > 2$. When we get an observation in this region, the commodity costs are

$$g_0(y,\pi) = \int_1^2 e^x \, dx = e^2 - e$$
$$g_1(y,\pi) = \int_0^1 e^x \, dx = e - 1$$

  and it is easy to see that $g_1(y,\pi) < g_0(y,\pi)$. Hence we decide $\mathcal{H}_1$ when we get observations in this region.

So, it turns out that we can boil the whole N-P decision rule down to

$$\rho^{\mathsf{NP}}(y) = \begin{cases} 1 & y \geq \ln(2e-1) \\ 0 & \text{otherwise.} \end{cases}$$

5. 4 points. Suppose you have n independent and identically distributed (i.i.d.) observations, each taking on the values 1 and 0 with probabilities p and $1-p$, respectively. Find a uniformly most powerful decision rule with false positive probability $\alpha = 2^{-n}$ for the hypothesis pair

$$\mathcal{H}_0 : p = \frac{1}{2}$$
$$\mathcal{H}_1 : p > \frac{1}{2}.$$

Solution: Let's let the observation y be the total number of ones observed from the n i.i.d. observations. Then $\mathcal{Y} = \{0, \ldots, n\}$. Hypothesis $\mathcal{H}_0$ has only one state in it and the conditional distribution associated with this state is

$$p_0(y = k) = \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \binom{n}{k} 2^{-n}.$$

Hypothesis $\mathcal{H}_1$ has an uncountably infinite number of states associated with it. Lets pick one state $x_1 : p = p_1 > 1/2$ and associate this one state with the simple hypothesis $\mathcal{H}_1'$. The conditional distribution associated with this state is

$$p_1(y = k) = \binom{n}{k} p_1^k (1 - p_1)^{n-k}$$

Testing between $\mathcal{H}_0$ and $\mathcal{H}_1'$ is a simple binary hypothesis testing problem, so we can use the standard N-P template for our decision rule between these two simple hypotheses, i.e.

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } L(y) > v \\ \gamma & \text{if } L(y) = v \\ 0 & \text{if } L(y) < v \end{cases}$$

where

$$L(y) = \frac{p_1(y = k)}{p_0(y = k)} = \frac{p_1^k (1 - p_1)^{n-k}}{2^{-n}} = 2^n p_1^k (1 - p_1)^{n-k}$$

Note that $L(y)$ is monotonically increasing in k as you would expect since it is more likely that $p = p_1 > 1/2$ than $p = 1/2$ if you observe more ones. Hence, the condition $L(y) > v$ can be equivalently stated as $y > \tau$. We just need to find τ and γ to satisfy the false positive probability constraint and confirm that the critical region does not depend on p_1.

To find τ and γ, we can try setting $\tau = n$. The observation y is never larger than τ but, conditioned on x_0, the observation y is equal to τ with probability

$$p_0(y = \tau = n) = \binom{n}{n} 2^{-n} = 2^{-n}.$$

The false positive probability in this problem is

$$P_{\mathsf{fp}} = \sum_{k > \tau} p_0(y = k) + \gamma p_0(y = \tau) = \alpha$$

where it is given in the problem description that $\alpha = 2^{-n}$. In this case, $\sum_{k>\tau} p_0(y = k) = 0$ and $p_0(y = \tau)$, hence if we set $\gamma = 1$, we get exactly the desired false positive probability. In other words, the $\alpha = 2^{-n}$ size decision rule for deciding between $\mathcal{H}_0$ and $\mathcal{H}_1'$ is

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } y = n \\ 0 & \text{if } y < n \end{cases}$$

Note that the critical region $\mathcal{Y}_1 = \{n\}$ does not depend on p_1. Hence this is also a UMP decision rule of size $\alpha = 2^{-n}$ for deciding between $\mathcal{H}_0$ and the composite hypothesis $\mathcal{H}_1$.

6. 4 points. Kay II: 6.21

Solution: The state in this problem is $x = \sigma^2$. Since the observations are i.i.d. zero-mean Gaussian, the joint conditional density can be written as

$$p_x(y) = \frac{1}{(2\pi x)^{N/2}} e^{\frac{-\sum_{n=0}^{N-1} y^2[n]}{2x}} = \frac{1}{(2\pi x)^{N/2}} e^{\frac{-y^\top y}{2x}}.$$

The LMP decision rule is found by computing

$$L'_{\lambda_0}(y) := \frac{d}{dx} L_{x/\lambda_0}(y)|_{x=\lambda_0} = \frac{\frac{d}{dx} p_x(y)|_{x=\lambda_0}}{p_{\lambda_0}(y)}$$

where $\lambda_0 = \sigma_0^2$ in this problem. We can compute

$$\begin{aligned} \frac{\frac{d}{dx} p_x(y)|_{x=\lambda_0}}{p_{\lambda_0}(y)} &= \frac{\frac{d}{dx} \frac{1}{(2\pi x)^{N/2}} e^{\frac{-y^\top y}{2x}}|_{x=\sigma_0^2}}{\frac{1}{(2\pi \sigma_0^2)^{N/2}} e^{\frac{-y^\top y}{2\sigma_0^2}}} \\ &= \frac{-N}{2\sigma_0^2} + \frac{y^\top y}{2\sigma_0^4}. \end{aligned}$$

The LMP decision rule ρ takes the form

$$\rho(y) = \begin{cases} 1 & L'_{\lambda_0}(y) > \tau \\ \gamma & L'_{\lambda_0}(y) = \tau \\ 0 & L'_{\lambda_0}(y) < \tau. \end{cases}$$

Note that $\frac{-N}{2\sigma_0^2}$ and $\frac{1}{2\sigma_0^4}$ are just constants (the latter of which is positive) and the distribution of $Y^\top Y$ is continuous, hence our decision rule can be simplified to

$$\rho(y) = \begin{cases} 1 & y^\top y \geq \tau' \\ 0 & y^\top y < \tau' \end{cases} = \begin{cases} 1 & \sum_{n=0}^{N-1} y^2[n] \geq \tau' \\ 0 & \sum_{n=0}^{N-1} y^2[n] < \tau' \end{cases}$$

where τ' is selected to satisfy the false positive probability constraint. This should make intuitive sense since $\sum_{n=0}^{N-1} y^2[n]$ is proportional to the sample variance of the observations and it is reasonable to select $\mathcal{H}_1$ when the sample variance exceeds some threshold. Finding the threshold must be done numerically since $Z = \sum_{n=0}^{N-1} Y^2[n]$ is chi-squared.

The easiest way to see that a UMP decision rule exists is to show that this family of densities has a monotone likelihood ratio for any $x_1 = \sigma_1^2 > \sigma_0^2 = x_0$. To see this, first note that the conditional distributions are all distinct for all $x \in \mathcal{X} = (0, \infty)$. In other words, there aren't two different values for $x = \sigma^2$ that give the same distribution. Now form the likelihood ratio

$$L_{x_1/x_0}(y) = \frac{p_{x_1}(y)}{p_{x_0}(y)} = \frac{\frac{1}{(2\pi x_1)^{N/2}} e^{\frac{-y^\top y}{2x_1}}}{\frac{1}{(2\pi x_0)^{N/2}} e^{\frac{-y^\top y}{2x_0}}} = \left(\frac{x_0}{x_1}\right)^{N/2} e^{-y^\top y\left(\frac{1}{2x_1} - \frac{1}{2x_0}\right)}.$$

Note that $\left(\frac{x_0}{x_1}\right)^{N/2}$ is a positive constant and $\frac{1}{2x_1} - \frac{1}{2x_0} = \frac{x_0 - x_1}{2x_0 x_1}$ is a negative constant (both are not functions of y). So let's rewrite this last expression as

$$L_{x_1/x_0}(y) = a e^{b y^\top y}.$$

where $a > 0$ and $b > 0$ are constants that are only functions of x_0, x_1, and N. Let $T(y) = y^\top y$. Then clearly $L_{x_1/x_0}(y)$ is non decreasing in $T(y)$. Hence, we can conclude that $p_x(y)$ for $x \in \mathcal{X} = (0, \infty)$ has a monotone likelihood ratio, the UMP decision rule exists, and will have the form

$$\rho(y) = \begin{cases} 1 & y^\top y \geq \tau' \\ 0 & y^\top y < \tau' \end{cases}$$

which is the same form as the LMP decision rule. Since τ' is selected to satisfy the false positive probability constraint, τ' will be the same for the UMP and LMP decision rules. Hence the UMP and LMP decision rules are identical in this problem.