ECE531 Midterm exam solution

1. (a) We will use the usual convention that $\mathcal{H}_0$ denotes the absence of something and $\mathcal{H}_1$ denotes the presence.

States: x_0 : patient did not take drug
x_1 : patient did take drug

Hypotheses: $\mathcal{H}_0 = \{x_0\}$
$\mathcal{H}_1 = \{x_1\}$

Observations: We perform one test, so the observation must come from the set $\mathcal{Y} = \{\underset{y_0}{P}, \underset{y_1}{I}, \underset{y_2}{NP}\}$
where P = "drug present"
I = "inconclusive"
NP = "drug not present"

Conditional distributions:

$$P_{x_0}(y) = \begin{cases} 0.1 & \text{if } y = P = y_0 \\ 0.2 & \text{if } y = I = y_1 \\ 0.7 & \text{if } y = NP = y_2 \end{cases}$$

$$P_{x_1}(y) = \begin{cases} 0.8 & \text{if } y = P = y_0 \\ 0.15 & \text{if } y = I = y_1 \\ 0.05 & \text{if } y = NP = y_2 \end{cases}$$

this is a simple binary hypothesis testing problem

There are 3 possible observations, so there are $2^3 = 8$ possible deterministic decision rules

$$\mathcal{D} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

(b) There are really only a few "good" deterministic decision rules that form the pareto-optimal tradeoff surface.

D_1 = always decide $\mathcal{H}_0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

D_2 = decide $\mathcal{H}_0$ unless $y = P \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

D_3 = decide $\mathcal{H}_0$ unless $y = P$ or $y = I \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

D_4 = always decide $\mathcal{H}_1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

under D_1: $R_0(D_1) = 0$ and $R_1(D_1) = 1$ $\quad \left(\text{UCA} \Rightarrow C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$

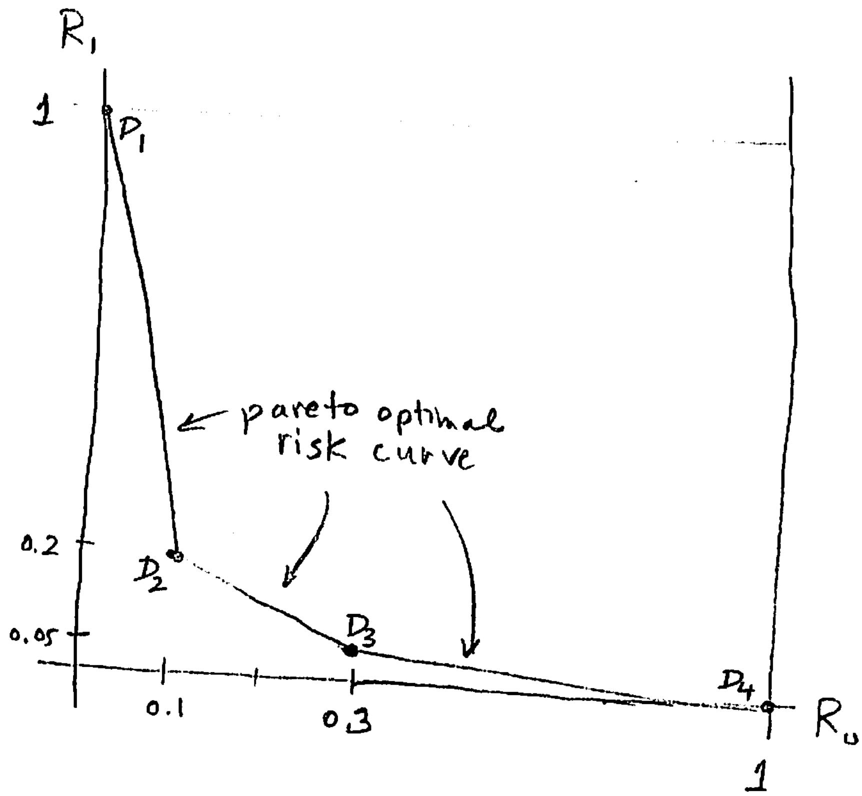
under D_2: $R_0(D_2) = c_0^T D_2 P_0 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix} = 0.1$

$$R_1(D_2) = c_1^T D_2 P_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.15 \\ 0.05 \end{bmatrix} = 0.2$$

under D_3: $R_0(D_3) = c_0^T D_3 P_0 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix} = 0.3$

$$R_1(D_3) = c_1^T D_3 P_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.15 \\ 0.05 \end{bmatrix} = 0.05$$

under D_4: $R_0(D_4) = 1$ and $R_1(D_4) = 0$



c)

We can see from the Pareto-optimal risk curve that D_2 provides the desired false positive probability and, since it is on the pareto-optimal risk curve, it is a N-P decision rule. No randomization is necessary.

Hence

$$\boxed{\rho^{NP}(y) = \begin{cases} 1 & y = \text{"P"} \\ 0 & y = \text{"NP"} \text{ or } y = \text{"I"} \end{cases}}$$

The probability of detection is simply $1 - R_1(D_2)$

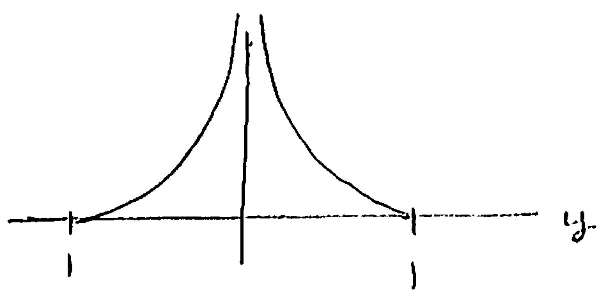
$$\boxed{P_D = 0.8}$$

2. a) This is a simple binary HT problem, so

$$\rho^{NP}(y) = \begin{cases} 1 & p_1(y) > v\, p_0(y) \\ \gamma & p_1(y) = v\, p_0(y) \\ 0 & p_1(y) < v\, p_0(y) \end{cases}$$

where $y \in \mathcal{Y} = [-1, 1]$ and v is selected to satisfy the false positive probability constraint α.

$$L(y) = \frac{p_1(y)}{p_0(y)} = \frac{1-|y|}{|y|} = \frac{1}{|y|} - 1 \quad \text{for } y \in \mathcal{Y}$$



So

$$p_1(y) > p_0(y) \iff L(y) > v \iff |y| < \tau$$

hence

$$\rho^{NP}(y) = \begin{cases} 1 & |y| \le \tau \\ 0 & |y| > \tau \end{cases}$$

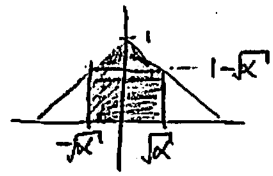
Where we have discarded the randomization since Prob $(|y| = \tau) = 0$. We just need to find τ to satisfy the false positive probability constraint.

$$P_{fp} = \text{Prob}\{|y| \le \tau \,;\, x = x_0\} = \int_{-\tau}^{\tau} p_0(y)\, dy = \frac{\tau^2}{2} + \frac{\tau^2}{2} = \tau^2$$

hence we should set $\tau^2 = \alpha$ The final form of the decision rule is then

$$\boxed{\rho^{NP}(y) = \begin{cases} 1 & |y| \le \sqrt{\alpha} \\ 0 & |y| > \sqrt{\alpha} \end{cases}}$$

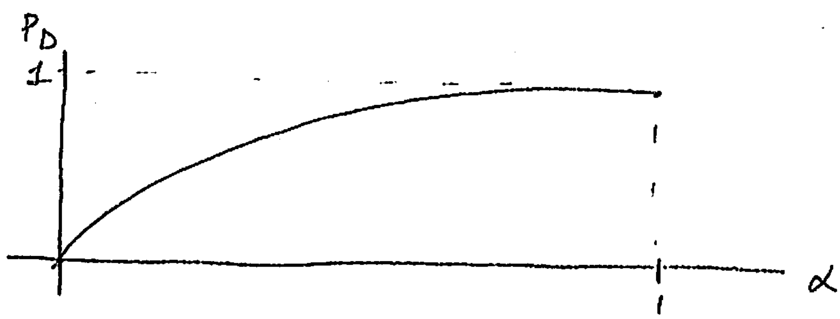
(5)

(b) $P_D = \int_{-\sqrt{\alpha}}^{\sqrt{\alpha}} p_1(y)\,dy =$ 

$$= 2\sqrt{\alpha}\,(1-\sqrt{\alpha}) + \tfrac{1}{2}\,2\sqrt{\alpha}\,\sqrt{\alpha}$$
$$= 2\sqrt{\alpha} - 2\alpha + \alpha$$
$$= 2\sqrt{\alpha} - \alpha$$

check: $\alpha = 0 \Rightarrow P_D = 0$ ✓

$\alpha = 1 \Rightarrow P_D = 2 - 1 = 1$ ✓



(c) simple binary Bayes HT problem

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } L(y) > \tau \\ 0/1 & \text{if } L(y) = \tau \\ 0 & \text{if } L(y) < \tau \end{cases}$$

where $\tau = \dfrac{\pi_0\,(C_{10} - C_{00})}{\pi_1\,(C_{01} - C_{11})} = \dfrac{\pi_0}{1-\pi_0}$

$$L(y) = \frac{1-|y|}{|y|} > \frac{\pi_0}{1-\pi_0} \iff (1-|y|)(1-\pi_0) > \pi_0|y|$$
$$\iff 1-\pi_0 - |y| + \cancel{\pi_0|y|} > \cancel{\pi_0|y|}$$
$$\iff |y| < 1-\pi_0$$

Hence

$$\boxed{\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } |y| < 1-\pi_0 \\ 1/0 & \text{if } |y| = 1-\pi_0 \\ 0 & \text{if } |y| > 1-\pi_0 \end{cases}}$$

← similar form to ρ^{NP}, just the threshold is different.

3. (a) let $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$

A UMP decision rule exists iff and only if the critical region for the simple binary HT

i) $\mathcal{H}_0 : x_0$ vs. $\mathcal{H}_1 : x_1$

is the same as

ii) $\mathcal{H}_0 : x_0$ vs. $\mathcal{H}_1 = x_2$

For $\mathcal{H}_1 = x_i$, $i \in \{1,2\}$,

The N-P decision rule is $\rho^{NP}(y) = \begin{cases} 1 & L(y) > v \\ \gamma & L(y) = v \\ 0 & L(y) < v \end{cases}$

where $L(y) = \dfrac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-i)^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}} = e^{\frac{y^2-(y-i)^2}{2}}$

$$\ln(L(y)) = \frac{1}{2}\left(y^2 - (y^2 - 2iy + i^2)\right) = \frac{1}{2}(2iy - i^2)$$

hence $L(y) > v \iff \frac{1}{2}(2iy - i^2) > \ln(v)$

$$\iff y > \frac{2\ln(v) + i^2}{2i} = v'$$

where v' is selected to satisfy the false positive probability constraint.

$$P_{fp} = \int_{v'}^{\infty} p_0(y)\,dy = Q(v') = \alpha$$

hence $v' = \dfrac{2\ln(v)+i}{2i} = Q^{-1}(\alpha)$ and

$$\boxed{\rho^{NP}(y) = \begin{cases} 1 & y \geq Q^{-1}(\alpha) \\ 0 & y < Q^{-1}(\alpha) \end{cases}}$$

where we have removed the randomization since $\text{Prob}(y = Q^{-1}(\alpha)) = 0$

The critical region doesn't depend on i, so this decision rule is <u>UMP</u>. $\Rightarrow$ UMP exists and we found it

(b) $\text{Prob}(x = x_0) = \text{Prob}(x = x_1) = \text{Prob}(x = x_2) = \frac{1}{3}$

Composite binary Bayes HT.

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } J(y) > 1 \\ 1/0 & \text{if } J(y) = 1 \\ 0 & \text{if } J(y) < 1 \end{cases}$$

where
$$J(y) = \frac{\sum_j \pi_j \, C_{0j} \, P_j(y)}{\sum_j \pi_j \, C_{1j} \, P_j(y)}$$

$$\left.\begin{array}{l} C_{0j} = \begin{cases} 0 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \text{ or } j = 2 \end{cases} \\ \\ C_{1j} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j = 1 \text{ or } j = 2 \end{cases} \end{array}\right\} \text{UCA}$$

Hence
$$J(y) = \frac{\frac{1}{3}\left(P_1(y) + P_2(y)\right)}{\frac{1}{3} P_0(y)}$$

Where $P_i(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-i)^2}{2}}$

Simplifying...

$$J(y) = \frac{e^{-\frac{(y-1)^2}{2}} + e^{-\frac{(y-2)^2}{2}}}{e^{-y^2/2}} = e^{\frac{2y-1}{2}} + e^{\frac{4y-4}{2}}$$

Hence, our decision rule requires us to compare

$$e^{-2} e^{2y} + e^{-1/2} e^{y} \gtreqless 1$$

let $z = e^y$, then $e^{-2} z^2 + e^{-1/2} z - 1 \gtreqless 0$

quadratic equation

roots : $\dfrac{-e^{-1/2} \pm \sqrt{(e^{-1/2})^2 + 4e^{-2}}}{2e^{-2}} \Rightarrow z = -5.7637$ and $z = 1.2820$

But $z = e^y$, so $z > 0$, hence the only root that makes sense is $z = 1.2820$

$\Rightarrow y = \ln(z) = 0.2484$

Hence, our decision rule is

$$\boxed{\delta^{B\pi}(y) = \begin{cases} 1 & y \geq 0.2484 \\ 1/0 & y = 0.2484 \\ 0 & y < 0.2484 \end{cases}}$$

The threshold here is significantly left of the threshold for testing

$\mathcal{H}_0 : Y \sim N(0,1)$

vs. $\mathcal{H}_1 : Y \sim N(1,1)$

This is because the composite HT problem assumes all 3 states are equally likely, but $\mathcal{H}_0$ is inherently less likely than $\mathcal{H}_1$ when $\mathcal{H}_1 = \{1, 2\}$.

end