ECE531 Screencast 1.3: Unbiased Estimators

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Uniformly Best Estimator? (part 1)

Consider the squared-error cost assignment and a scalar non-random parameter θ. The average cost is

$$R_\theta(\hat{\theta}) \quad = \quad \mathrm{E}_\theta\left[(\theta - \hat{\theta}(Y))^2\right]$$

where the expectation is taken over the observations.

Question: Is it possible to find a "uniformly best" estimator that minimizes $R_\theta(\hat{\theta})$ for all θ?

Uniformly Best Estimator? (part 2)

Suppose you receive one observation

$$Y = \theta + W$$

where $\theta \in \mathbb{R}$ and $W \sim \mathcal{N}(0,1)$. Suppose further that your estimator is

$$\hat{\theta}(y) = ay$$

where a is a scalar parameter that you will specify to minimize $R_\theta(\hat{\theta})$. We can calculate

$$\begin{aligned} R_\theta(\hat{\theta}) &= \mathrm{E}_\theta\left[(\theta - aY)^2\right] \\ &= \theta^2 - 2a\theta\mathrm{E}[Y] + a^2\mathrm{E}[Y^2] \\ &= \theta^2 - 2a\theta^2 + a^2(\theta^2+1) \end{aligned}$$

How can we find the value of a that minimizes this?

Uniformly Best Estimator? (part 3)

To find the value of a that minimizes

$$R_\theta(\hat{\theta}) \quad = \quad \theta^2 - 2a\theta^2 + a^2(\theta^2 + 1)$$

we can take a derivative with respect to a, set it to zero, and solve for a to get

$$a = \frac{\theta^2}{\theta^2 + 1}.$$

You can easily verify this value of a is in fact a minimum. Hence

$$\hat{\theta}(y) = \frac{\theta^2}{\theta^2 + 1} y$$

is the estimator that minimizes $R_\theta(\hat{\theta})$ over all θ.

What is the problem with this result?

Our Approach: Consider Only Unbiased Estimators

Since "uniformly best" non-random parameter estimators are unlikely to exist in most cases, we will consider only the class of **unbiased estimators**.

Definition

An estimator $\hat{\theta}(y)$ is unbiased if

$$\mathrm{E}_\theta\left[\hat{\theta}(Y)\right] \quad = \quad \theta$$

for all $\theta \in \Lambda$.

Remarks:

- This class excludes trivial estimators like $\hat{\theta}(y) \equiv \theta_0$ (constant).
- Under the squared-error cost assignment, the average cost of estimators in this class

$$R_\theta(\hat{\theta}) = \mathrm{E}_\theta\left[(\theta - \hat{\theta}(Y))^2\right] = \mathrm{var}_\theta\left[\hat{\theta}(Y)\right]$$

- The goal: **find an unbiased estimator with minimum variance**.