ECE531 Screencast 1.4: Minimum Variance Unbiased Estimators

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Minimum Variance Unbiased Estimators

Definition

A minimum-variance unbiased estimator $\hat{\theta}_{\mathsf{mvu}}(y)$ is an unbiased estimator satisfying

$$\hat{\theta}_{\mathsf{mvu}}(y) = \arg \min_{\hat{\theta}(y) \in \Omega} R_\theta(\hat{\theta}(y))$$

for all $\theta \in \Lambda$ where Ω is the set of all unbiased estimators.

Remarks:

- Finding an MVU estimator is a multi-objective optimization problem. You have to find **one** estimator to minimize the variance at all $\theta \in \Lambda$.
- The estimator can not be a function of θ.
- MVU estimators do not always exist (see Example 2.3 in Kay I).
- We will see, however, that lots of problems do yield MVU estimators.

Example: Estimating a Constant in White Gaussian Noise

Suppose we have random observations given by

$$Y_k = \theta + W_k \qquad k = 0, \dots, n-1$$

where $W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$ with $\theta \in \mathbb{R}$.

Suppose we go with an estimator that performs the sample mean:

$$\hat{\theta}(y) = \frac{1}{n} \sum_{k=0}^{n-1} y_k$$

- Is this estimator unbiased? Yes (easy to check).
- Is this estimator MVU? The variance of the estimator can be calculated as $\text{var}_\theta \left[\hat{\theta}(Y) \right] = \frac{\sigma^2}{n}$. But answering the question as to whether this estimator is MVU or not will require more work.

Example: Estimating Mean and Variance

Suppose we have random observations given by

$$Y_k \quad = \quad \theta_1 + W_k \qquad k = 0, \dots, n-1$$

where $W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \theta_2)$. Note that both θ_1 and $\theta_2 > 0$ are unknown. This is called a vector parameter estimation problem.

We already know an unbiased estimator for θ_1: the sample mean $\hat{\theta}_1(y) = \frac{1}{n}\sum_{k=0}^{n-1} y_k$. This estimator is still valid in this case because the sample mean does not depend on any unknown parameters.

How about this estimator for θ_2 (sample variance):

$$\hat{\theta}_2(y) \quad = \quad \frac{1}{n}\sum_{k=0}^{n-1}(y_k - \hat{\theta}_1(y))^2$$

- Is this estimator unbiased?
- Is this estimator MVU?

Example: Estimating Mean and Variance

To check if the estimator for θ_2 is unbiased, we can calculate the mean:

$$\begin{aligned}
\mathrm{E}_\theta[\hat{\theta}_2(Y)] &= \mathrm{E}_\theta\left[\frac{1}{n}\sum_{\ell=0}^{n-1}\left(Y_\ell - \hat{\theta}_1(Y)\right)^2\right] \\
&= \frac{1}{n}\sum_{\ell=0}^{n-1}\mathrm{E}_\theta\left[\left(Y_\ell - \hat{\theta}_1(Y)\right)^2\right] \\
&= \frac{1}{n}\sum_{\ell=0}^{n-1}\mathrm{E}_\theta\left[\left(Y_\ell - \left(\frac{1}{n}\sum_{k=0}^{n-1}Y_k\right)\right)^2\right] \\
&= \frac{1}{n}\sum_{\ell=0}^{n-1}\mathrm{E}_\theta\left[\left(\theta_1 + W_\ell - \left(\frac{1}{n}\sum_{k=0}^{n-1}(\theta_1 + W_k)\right)\right)^2\right] \\
&= \frac{1}{n^3}\sum_{\ell=0}^{n-1}\mathrm{E}_\theta\left[\left((n-1)W_\ell - \sum_{\substack{k=0\\k\neq\ell}}^{n-1}W_k\right)^2\right] \\
&= \frac{1}{n^3}\sum_{\ell=0}^{n-1}\left[(n-1)^2 + n - 1\right]\theta_2 = \frac{n-1}{n}\theta_2
\end{aligned}$$

Example: Estimating Mean and Variance

Since

$$\mathrm{E}_\theta[\hat{\theta}_2(Y)] = \frac{n-1}{n}\theta_2 \neq \theta_2$$

this estimator is biased.

We can use this result to make an unbiased estimator for θ_2, however:

$$\hat{\theta}_2(y) \quad = \quad \frac{1}{n-1}\sum_{k=0}^{n-1}(y_k - \hat{\theta}_1(y))^2$$

You can confirm this estimator is unbiased by computing the mean $\mathrm{E}_\theta[\hat{\theta}_2(Y)]$ as we did before.

Finding MVU Estimators

There is no "plug-and-chug" method that you can always follow to find an MVU estimator. We will cover two common approaches that work in many cases:

1. The Cramer-Rao lower bound (Kay I: Chapters 3-4)
    - Guess at a good estimator and check if the variance achieves the theoretical minimum (CRLB).
    - Take advantage of special cases, e.g. linear model
2. The Rao-Blackwell-Lehmann-Sheffe (RBLS) theorem (Kay I: Chapter 5)
    - Finding a "complete sufficient statistic" for the observations.
    - Performing a conditional expectation to get the MVU estimator.