

# ECE531 Screencast 1.5: Kay I: Problem 2.2

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# Problem Statement

**2.2** consider the data  $\{x[0], x[1], \dots, x[N-1]\}$ , where each sample is distributed as  $\mathcal{U}(0, \theta)$  and the samples are i.i.d. Can you find an unbiased estimator for  $\theta$ ? The range of  $\theta$  is  $0 < \theta < \infty$ .

The problem here is to estimate the upper limit on the distribution from which the samples  $x[n]$  are drawn. For example, suppose you saw observations like this: 0.8574, 0.0116, 0.9364, 0.3227, 1.9273, 1.1459, 1.5776, 0.8781, 1.1251, 1.2688. What would you guess  $\theta$  is?

A bad guess would be  $\theta = 1$ .

You might be tempted to try

$$\hat{\theta}(x) = \max_n x[n],$$

but this is clearly biased for finite  $N$ .

# A solution

Let's look at the sample mean in this case:

$$g(x) = \text{sample mean} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

This is clearly going to be biased, but let's proceed anyhow. We can compute the mean of our sample mean

$$\begin{aligned} \mathbf{E}_{\theta}[g(x)] &= \mathbf{E}_{\theta} \left[ \frac{1}{N} \sum_{n=0}^{N-1} X[n] \right] \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{E}_{\theta} [X[n]] \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \frac{\theta}{2} = \frac{\theta}{2} \end{aligned}$$

Hence an unbiased estimator could be  $\hat{\theta}_a(x) = 2g(x) = \frac{2}{N} \sum_{n=0}^{N-1} x[n]$ .

# Uniqueness of Unbiased Estimators?

This is not the only possible unbiased estimator. Another unbiased estimator is

$$\hat{\theta}_b(x) = 2x[0].$$

You can easily confirm this estimator is unbiased.

In general, unbiased estimators are not unique.

## Which estimator is better?

While we do not have the tools yet to determine if either estimator is MVU, we can certainly determine which estimator is better under the squared error cost assignment.

The variance of the first estimator is

$$\text{var}(\hat{\theta}_a(X)) = \text{var} \left\{ \frac{2}{N} \sum_{n=0}^{N-1} X[n] \right\} = \frac{4}{N^2} \text{var}(X[n]) = \frac{4}{N^2} \cdot \frac{\theta^2}{12}$$

and the variance of the second estimator is

$$\text{var}(\hat{\theta}_b(X)) = \text{var} \{2X[0]\} = 4\text{var}(X[0]) = 4 \cdot \frac{\theta^2}{12}.$$

Clearly the first estimator is better if  $N > 1$ . This makes intuitive sense because the first estimator uses more information in forming its estimate than the second estimator.