ECE531 Screencast 1.6: Kay I: Problem 2.9

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Problem Statement

2.9 This problem illustrates what happens to an unbiased estimator when it undergoes a nonlinear transformation. In example 2.1, if we choose to estimate the unknown parameter $\theta = A^2$ by

$$\hat{\theta} = \left(\frac{1}{N} \sum_{n=0}^{N-1} x[n] \right)^2$$

can we say that the estimator is unbiased? What happens as $N \to \infty$?

Example 2.1 is about estimating a DC level A in white Gaussian noise. We know the sample mean estimator is an unbiased estimator of A. This problem is about estimating A^2, however. The question is can we just square the sample mean estimate to get an unbiased estimate of A^2?

A solution

To answer the question about bias, recall that $X[n] = A + W[n]$ with $W[n] \sim \mathcal{N}(0, \sigma^2)$ and let's compute the mean of our estimator:

$$\begin{aligned}
\mathrm{E}_\theta(\hat{\theta}(X)) &= \mathrm{E}_\theta\left(\left(\frac{1}{N}\sum_{n=0}^{N-1} X[n]\right)^2\right) \\
&= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \mathrm{E}_\theta(X[n]X[m]) \\
&= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \mathrm{E}_\theta((A + W[n])(A + W[m])) \\
&= \frac{1}{N^2}\left[N(A^2 + \sigma^2) + N(N-1)A^2\right] \\
&= A^2 + \frac{\sigma^2}{N}
\end{aligned}$$

so this is clearly biased if $\sigma^2 > 0$ and $N < \infty$.

Remarks

Using this result, we could form an unbiased estimator

$$\hat{\theta} = \left(\frac{1}{N} \sum_{n=0}^{N-1} x[n] \right)^2 - \frac{\sigma^2}{N}.$$

This estimator is valid because σ^2 and N are known.

Finally, what happens as $N \to \infty$? We see from the previous result that the bias vanishes. We call such an estimator "asymptotically unbiased". Even though it is biased for finite N, the bias vanishes as $N \to \infty$.