ECE531 Screencast 10.2: Binary Detection in Additive White Gaussian Noise

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Binary Detection in Additive White Gaussian Noise

Problem setup:

- We have two known discrete-time signals $s_0, s_1 \in \mathbb{R}^n$ that are observed in additive white Gaussian noise.
- A signal $x \in \{s_0, s_1\}$ is transmitted and we observe a realization $y \in \mathbb{R}^n$ of the random variable

$$Y = x + W$$

  where $W \sim \mathcal{N}(0, \sigma^2 I)$ is zero mean-additive Gaussian noise.
- We assume that the receiver knows the noise distribution $\mathcal{N}(0, \sigma^2 I)$.
- Given the observation y, we must decide whether s_0 or s_1 was transmitted.
- This is called a **coherent** detection problem because the signals s_0 and s_1 are deterministic and are completely known in advance to the detector/receiver.

Conditional Distributions and Decision Rules

Conditioned on $x = s_j$, we can write density of the vector observation as

$$p_j(y) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{(y-s_j)^\top (y-s_j)}{2\sigma^2}\right)$$

This coherent detection problem is just a simple binary HT problem. We know we are going to have decision rules of the form:

$$\rho(y) = \begin{cases} 1 & \text{if } L(y) > v \\ \gamma & \text{if } L(y) = v \\ 0 & \text{if } L(y) < v \end{cases}$$

Hence, we are going to have to compute the likelihood ratio...

Likelihood Ratio

$$L(y) = \frac{p_1(y)}{p_0(y)} = \exp\left(\frac{(y-s_0)^\top(y-s_0)-(y-s_1)^\top(y-s_1)}{2\sigma^2}\right)$$

It is more convenient to work with the log-likelihood ratio here. Let

$$\begin{aligned}\ell(y) &:= 2\sigma^2\ln(L(y)) \\ &= (y-s_0)^\top(y-s_0)-(y-s_1)^\top(y-s_1)\end{aligned}$$

Then we can write the decision rule template as

$$\rho(y) = \begin{cases} 1 & \text{if } \ell(y) \geq 2\sigma^2 \ln v \\ 0 & \text{if } \ell(y) < 2\sigma^2 \ln v \end{cases}$$

Note that we dropped the case $\rho(y) = \gamma$ because randomization will not be needed here ($\ell(y) = 2\sigma^2 \ln v$ with probability zero).

Correlation Form of the Decision Rule

We can write the statistic used in our decision rule as

$$\begin{aligned}\ell(y) &= (y-s_0)^\top(y-s_0)-(y-s_1)^\top(y-s_1)\\ &= \|y\|^2 - 2s_0^\top y + \|s_0\|^2 - \left(\|y\|^2 - 2s_1^\top y + \|s_1\|^2\right)\\ &= 2(s_1-s_0)^\top y + ||s_0||^2 - ||s_1||^2\end{aligned}$$

Hence, our decision rule template can be further simplified as

$$\rho(y) = \begin{cases} 1 & (s_1-s_0)^\top y \geq \sigma^2 \ln v + \frac{1}{2}(||s_1||^2 - ||s_0||^2) \\ 0 & \qquad\qquad\quad < \end{cases} \tag{1}$$

All that remains is to specify v. Equivalently, since σ^2, $\|s_0\|^2$ and $\|s_1\|^2$ are all known, we can also just specify $v' = \sigma^2 \ln v + \frac{1}{2}(||s_1||^2 - ||s_0||^2)$.

Statistics of the Decision Variable $Z = (s_1 - s_0)^\top Y$

Our decision rule is based on a comparison of $z = (s_1 - s_0)^\top y$ with some threshold v'. Let $s := s_1 - s_0$.

- Note that $z = s^\top y$ is the inner product (or deterministic correlation) between the observation y and the signal difference vector s.
- When $x = s_j$, the observation $Y \sim \mathcal{N}(s_j, \sigma^2 I)$.
- When $x = s_j$, how is $Z = s^\top Y$ distributed?
  - Given $x = s_j$, $Z \sim \mathcal{N}(\mu_j, \sigma_Z^2)$ is a Gaussian random variable.

$$
\begin{aligned}
\mu_j &:= \mathrm{E}[Z \,|\, x = s_j] = \mathrm{E}[s^\top Y \,|\, x = s_j] = s^\top \mathrm{E}[Y \,|\, x = s_j] = s^\top s_j \\
&= (s_1 - s_0)^\top s_j \\
\sigma_Z^2 &:= \mathrm{E}[(Z - s^\top s_j)^2 \,|\, x = s_j] = \mathrm{E}[(s^\top Y - s^\top s_j)^2 \,|\, x = s_j] \\
&= \mathrm{E}[(s^\top (Y - s_j))^2 \,|\, x = s_j] = \mathrm{E}[(s^\top W)^2] \\
&= s^\top \mathrm{E}[W W^\top] s = \sigma^2 s^\top s = \sigma^2 \|s\|^2 \\
&= \sigma^2 (s_1 - s_0)^\top (s_1 - s_0).
\end{aligned}
$$

Neyman-Pearson and Bayes Decision Rules

In both cases, the decision rule is of the form (with $z = (s_1 - s_0)^\top y$)

$$\rho(y) \quad = \quad \begin{cases} 1 & z \quad \geq v' \\ 0 & \quad\quad < \end{cases}$$

Neyman-Pearson: A false positive occurs if $x = s_0$ and $Z \geq v'$. Given $x = s_0$, the decision variable $Z \sim \mathcal{N}(s^\top s_0, \sigma^2 \|s\|^2)$. Hence

$$P_{\mathsf{fp}} = Q\left(\frac{v' - s^\top s_0}{\sigma \|s\|}\right) \leq \alpha$$

Setting this equal to α yields $v' = \sigma \|s\| Q^{-1}(\alpha) + s^\top s_0$. The probability of detection is then $P_D = Q\left(\frac{v' - s^\top s_1}{\sigma \|s\|}\right)$.

Bayes: The Bayes detector requires the specification of a prior π_0 and a cost matrix C. The decision threshold is $v' = \sigma^2 \ln \tau + \frac{1}{2}(||s_1||^2 - ||s_0||^2)$ with

$$\tau \quad := \quad \frac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})}.$$