ECE531 Screencast 10.3: Binary Detection in Additive Correlated Gaussian Noise

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Binary Detection in Correlated Noise

Problem setup:

- We have two known discrete-time signals $s_0, s_1 \in \mathbb{R}^n$ that are observed in additive white Gaussian noise.
- A signal $x \in \{s_0, s_1\}$ is transmitted and we observe a realization $y \in \mathbb{R}^n$ of the random variable

$$Y = x + W$$

  where $W \sim \mathcal{N}(0, \Sigma)$ is zero mean-additive Gaussian noise.
- The positive definite noise covariance matrix Σ is defined as

$$\Sigma \;=\; \mathrm{E}[WW^\top]$$

- We assume that the receiver knows the noise distribution $\mathcal{N}(0, \Sigma)$.
- Given the observation y, we must decide whether s_0 or s_1 was transmitted.
- Like the case with AWGN, this is a simple binary HT problem.

Likelihood Ratio

$$
\begin{aligned}
L(y) &= \frac{p_1(y)}{p_0(y)} \\
&= \exp\left(\frac{(y-s_0)^\top \Sigma^{-1}(y-s_0) - (y-s_1)^\top \Sigma^{-1}(y-s_1)}{2}\right)
\end{aligned}
$$

Converting to a log-likelihood ratio, we can write

$$
\begin{aligned}
\ell(y) &:= 2\ln(L(y)) \\
&= (y-s_0)^\top \Sigma^{-1}(y-s_0) - (y-s_1)^\top \Sigma^{-1}(y-s_1)
\end{aligned}
$$

The decision rule template is similar to the AWGN case:

$$
\rho(y) = \begin{cases} 1 & \text{if } \ell(y) \geq 2\ln v \\ 0 & \text{if } \ell(y) < 2\ln v \end{cases}
$$

Decorrelation

Lemma

A real symmetric matrix P is positive definite if and only if there exists a nonsingular matrix S such that $P = S^\top S$.

Given P, how can we find S such that $P = S^\top S$?
Cholesky factorization (see Matlab function `chol`).

Since Σ is positive definite, then so is Σ^{-1}. Hence, we can write

$$\Sigma^{-1} = S^\top S$$

where S is invertible. Now let

$$\begin{aligned}
\bar{y} &= Sy \\
&= S(x + w) \\
&= \begin{cases} \bar{s}_0 + \bar{w} & \text{if } x = s_0 \\ \bar{s}_1 + \bar{w} & \text{if } x = s_1 \end{cases}
\end{aligned}$$

Decorrelation

Note that S just specifies a one-to-one coordinate transformation between $\mathbb{R}^n$ and $\mathbb{R}^n$. In this new coordinate system

$$\begin{aligned}
(y - s_j)^\top \Sigma^{-1}(y - s_j) &= (y - s_j)^\top S^\top S(y - s_j) \\
&= [S(y - s_j)]^\top S(y - s_j) \\
&= (\bar{y} - \bar{s}_j)^\top (\bar{y} - \bar{s}_j) \\
&= ||\bar{y} - \bar{s}_j||^2
\end{aligned}$$

The noise is still Gaussian after this transformation, of course, with

$$\begin{aligned}
\mathrm{E}[\bar{W}] &= \mathrm{E}[SW] = 0 \\
\mathrm{E}[\bar{W}\bar{W}^\top] &= \mathrm{E}[SWW^\top S^\top] = S\mathrm{E}[WW^\top]S^\top = S\Sigma S^\top = I
\end{aligned}$$

Hence, the coordinate transformation has decorrelated (whitened) the noise. After this decorrelation operation, we can just use our prior results for binary detection in AWGN.

Neyman-Pearson and Bayes Decision Rules

In both cases, the decision rule is of the form (with $\bar{z} = (\bar{s}_1 - \bar{s}_0)^\top \bar{y}$)

$$\rho(y) = \begin{cases} 1 & \bar{z} \geq v' \\ 0 & \quad < \end{cases}$$

Neyman-Pearson: A false positive occurs if $x = s_0$ and $\bar{Z} \geq v'$. Given $x = s_0$, the decision variable $\bar{Z} \sim \mathcal{N}(\bar{s}^\top \bar{s}_0, \|\bar{s}\|^2)$. Hence

$$P_{\mathsf{fp}} = Q\left(\frac{v' - \bar{s}^\top \bar{s}_0}{\|\bar{s}\|}\right) \leq \alpha$$

Setting this equal to α yields $v' = \|\bar{s}\| Q^{-1}(\alpha) + \bar{s}^\top \bar{s}_0$. The probability of detection is then $P_D = Q\left(\frac{v' - \bar{s}^\top \bar{s}_1}{\sigma \|\bar{s}\|}\right)$.

Bayes: The Bayes detector requires the specification of a prior π_0 and a cost matrix C. The decision threshold is $v' = \ln \tau + \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2)$ with

$$\tau := \frac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})}.$$