ECE531 Screencast 10.4: Minimum Error Probability Binary Signal Detection

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Minimum Probability of Error: Bayes Detection with UCA

Suppose we have a binary additive Gaussian noise detection problem with $x \in \{s_0, s_1\}$ and a given prior π_0.

The probability of making an incorrect decision is (Bayes risk with UCA)

$$\begin{aligned} P_e &= \pi_0 \mathrm{Prob}(\text{decide } \mathcal{H}_1 \,|\, x = s_0) + (1-\pi_0)\mathrm{Prob}(\text{decide } \mathcal{H}_0 \,|\, x = s_1) \\ &= \pi_0 \mathrm{Prob}\left(\bar{s}^\top \bar{Y} \geq v' \,|\, x = s_0\right) + (1-\pi_0)\mathrm{Prob}\left(\bar{s}^\top \bar{Y} < v' \,|\, x = s_1\right) \end{aligned}$$

where v' is the decision threshold. For $\mu_j := (\bar{s}_1 - \bar{s}_0)^\top \bar{s}_j$, we can write

$$P_e = \pi_0 Q\left(\frac{v' - \mu_0}{\|\bar{s}\|}\right) + (1-\pi_0) Q\left(\frac{\mu_1 - v'}{\|\bar{s}\|}\right)$$

and we know the decision threshold v' that minimizes P_e is

$$v' = \ln \frac{\pi_0}{1-\pi_0} + \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2)$$

Uniform Prior and Optimum Detection Threshold

Suppose that we have a uniform prior $\pi_0 = \pi_1 = \frac{1}{2}$ and that the optimum threshold for this prior, i.e. $v' = \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2)$, is used by the detector. Note that

$$v' = \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2) = \frac{(\bar{s}_1 - \bar{s}_0)^\top \bar{s}_0 + (\bar{s}_1 - \bar{s}_0)^\top \bar{s}_1}{2} = \frac{\mu_0 + \mu_1}{2}$$

Hence, the probability of error can be written as

$$P_e = \frac{1}{2} Q\left(\frac{\mu_1 - \mu_0}{2\|\bar{s}\|}\right) + \frac{1}{2} Q\left(\frac{\mu_1 - \mu_0}{2\|\bar{s}\|}\right) = Q\left(\frac{\mu_1 - \mu_0}{2\|\bar{s}\|}\right)$$

We can also write

$$\mu_1 - \mu_0 = (\bar{s}_1 - \bar{s}_0)^\top \bar{s}_1 - (\bar{s}_1 - \bar{s}_0)^\top \bar{s}_0 = \|\bar{s}_0\|^2 + \|\bar{s}_1\|^2 - 2\bar{s}_0^\top \bar{s}_1 = \|\bar{s}\|^2$$

Hence

$$P_e = Q\left(\frac{||\bar{s}||}{2}\right)$$

Optimum Signal Design for Minimum Error Probability

In communication problems, we often can choose our signal alphabet. Since $Q(x)$ is monotonically decreasing in x, we should choose s_0 and s_1 to maximize $\|\bar{s}\|$. To get an useful result, we require the power of each signal s_0 and s_1 to be upper bounded by B, i.e.

$$\frac{1}{n}||s_j||^2 \le B \qquad \Rightarrow \qquad \frac{1}{n}||s_0 - s_1||^2 \le 4B$$

Recall $\bar{s} = (Ss_1 - Ss_0)$ with $\Sigma^{-1} = S^\top S$. We can write

$$\|\bar{s}\|^2 = (Ss_1 - Ss_0)^\top (Ss_1 - Ss_0) = (s_1 - s_0)^\top \Sigma^{-1} (s_1 - s_0)$$

Let $\lambda_{max}(\Sigma^{-1})$ be the largest eigenvalue of the positive definite matrix Σ^{-1} and let ν be the eigenvector associated with this eigenvalue. Then

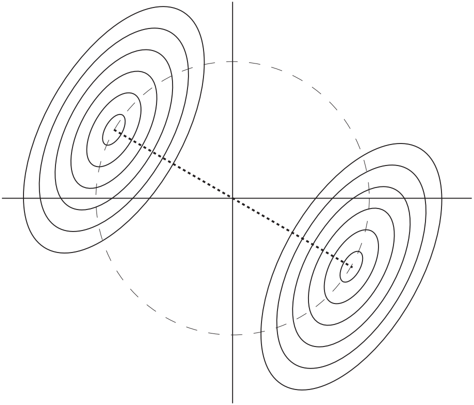
$$\|\bar{s}\|^2 \le \lambda_{max}(\Sigma^{-1})||s_1 - s_0||^2$$

with equality only if $s_1 - s_0 = \alpha\nu$ for some scalar α. Hence

$$(s_1 - s_0)^\top \Sigma^{-1} (s_1 - s_0) = ||\bar{s}||^2 \le 4nB\lambda_{max}(\Sigma^{-1}).$$

Optimum Signal Design: Geometric Interpretation

The probability of error is minimized when $s_1 - s_0$ is aligned with the eigenvector of $\Sigma^{-1} = 1/\lambda_{min}(\Sigma)$ corresponding to the maximum eigenvalue $\lambda_{max}(\Sigma^{-1})$ and arranged antipodally on the sphere of radius $\sqrt{nB}$.



Optimum Signal Design: Summary

- The probability of error is minimized when $s_1 - s_0$ is aligned with the eigenvector of Σ corresponding to the smallest eigenvalue $\lambda_{min}(\Sigma)$, i.e. $s_1 - s_0$ is aligned in the direction of least noise variance.
- The minimum achievable error probability with optimum signaling (and equiprobable signals) is then

$$P_e^* = Q\left(\frac{||\bar{s}||}{2}\right) = Q\left(\frac{\sqrt{4nB\lambda_{max}(\Sigma^{-1})}}{2}\right) = Q\left(\frac{\sqrt{nB}}{\sigma_{min}}\right)$$

where $\sigma_{min}^2 = \lambda_{min}(\Sigma)$.