ECE531 Screencast 10.5: M-ary Bayesian Detection

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M-ary Detection in Correlated Noise

Problem setup:

- We have M known discrete-time signals $s_0, \dots, s_{M-1} \in \mathbb{R}^n$ that are observed in additive white Gaussian noise.
- A signal $x \in \{s_0, \dots, s_{M-1}\}$ is transmitted and we observe a realization $y \in \mathbb{R}^n$ of the random variable

$$Y = x + W$$

  where $W \sim \mathcal{N}(0, \Sigma)$ is zero mean-additive Gaussian noise.
- The positive definite noise covariance matrix Σ is defined as

$$\Sigma \;=\; \mathrm{E}[WW^\top]$$

- We assume that the receiver knows the noise distribution $\mathcal{N}(0, \Sigma)$.
- Given the observation y, we must decide which signal was transmitted.
- This is a simple M-ary HT problem.
- We will only consider Bayesian detection in this case.

M-ary Bayesian Decision Rules

For a given prior $\pi = [\pi_0, \dots, \pi_{M-1}]$, hypotheses $\mathcal{H}_0, \dots, \mathcal{H}_{M-1}$ associated each with one of the possible signals, and costs C_{ij} for $i, j \in 0, \dots, M-1$, we have the deterministic Bayesian decision rule (infinite observation space):

$$\delta^{B\pi}(y) = \arg \min_{i \in \{0,\dots,M-1\}} \sum_{j=0}^{M-1} \pi_j C_{ij} p_j(y)$$

where

$$p_j(y) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{(y-s_j)^\top \Sigma^{-1}(y-s_j)}{2}\right)$$

This is the general form. M-ary detection does not reduce to simply computing a likelihood ratio unless $M = N = 2$.

M-ary Bayesian Decision Rules: Simplification

We have

$$p_j(y) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{(y-s_j)^\top \Sigma^{-1}(y-s_j)}{2}\right)$$

The term $\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} > 0$ is common to all of these $p_j(y)$, so we can ignore it in the minimization. Hence

$$\begin{aligned}\delta^{B\pi}(y) &= \arg\min_{i\in\{0,\dots,M-1\}} \sum_{j=0}^{M-1} \pi_j C_{ij} \exp\left(-\frac{(y-s_j)^\top \Sigma^{-1}(y-s_j)}{2}\right) \\ &= \arg\min_{i\in\{0,\dots,M-1\}} \sum_{j=0}^{M-1} \pi_j C_{ij} \exp\left(-\frac{1}{2}\|\bar{y}-\bar{s}_j\|^2\right)\end{aligned}$$

where $\Sigma^{-1} = S^\top S$, $\bar{y} = Sy$, and $\bar{s}_j = Ss_j$. Further simplification will require some assumptions about the prior and the costs...

M-ary Bayesian Decision Rules: Equiprobable with UCA

In the case of an M-ary communication system, we typically have equiprobable signals and assume the UCA so that the Bayes risk becomes the probability of error. In this case, we have $\pi_j = \frac{1}{M}$ and $C_{ij} = 0$ if $i = j$ and $C_{ij} = 1$ otherwise. Hence,

$$
\begin{aligned}
\delta^{B\pi}(y) &= \arg \min_{i \in \{0,\dots,M-1\}} \sum_{j=0}^{M-1} \pi_j C_{ij} \exp\left(-\frac{1}{2}\|\bar{y} - \bar{s}_j\|^2\right) \\
&= \arg \min_{i \in \{0,\dots,M-1\}} \sum_{j=0}^{M-1} (1 - \delta(i-j)) \exp\left(-\frac{1}{2}\|\bar{y} - \bar{s}_j\|^2\right) \\
&= \arg \max_{i \in \{0,\dots,M-1\}} \exp\left(-\frac{1}{2}\|\bar{y} - \bar{s}_i\|^2\right) \\
&= \arg \min_{i \in \{0,\dots,M-1\}} \|\bar{y} - \bar{s}_i\|^2
\end{aligned}
$$

This is the minimum distance detector. If the noise is correlated, the distance is measured between the decorrelated observation $\bar{y}$ and the decorrelated signal $\bar{s}_j$.

Minimum Distance Detection: Voronoi Regions