ECE531 Screencast 10.6: Deterministic Signal Detection Example

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Problem Setup: Kay vII Problem 4.6

It is desired to detect the known signal $s[k] = Ar^k$ for $k = 0, \dots, n-1$ in white Gaussian noise with variance σ^2 Find the Neyman-Pearson detector and its detection performance. Explain what happens as $n \to \infty$ for $0 < r < 1$, $r = 1$, and $r > 1$.

There are two hypotheses here:

$$\mathcal{H}_0 = \text{the signal is not present}$$
$$\mathcal{H}_1 = \text{the signal is present}$$

We can think of the states as

$$x_0 = \text{the signal is } s_0 = [0, \dots, 0]^\top$$
$$x_1 = \text{the signal is } s_1 = [A, Ar, \dots, Ar^{n-1}]^\top$$

We have two states and two hypotheses, so this is a simple binary hypothesis testing problem in AWGN.

Decision Rule Template

Let $y = [y_0, \dots, y_{n-1}]^\top$ be the observation. Since we have a detection problem in AWGN, the decision rule template can be written as

$$\rho(y) \quad = \quad \begin{cases} 1 & (s_1 - s_0)^\top y \quad \geq \sigma^2 \ln v + \frac{1}{2}(||s_1||^2 - ||s_0||^2) \\ 0 & \qquad\qquad\quad < \end{cases}$$

which, when $s_0 = [0, \dots, 0]^\top$ further simplifies to

$$\rho(y) \quad = \quad \begin{cases} 1 & s_1^\top y \quad \geq \sigma^2 \ln v + \frac{1}{2}||s_1||^2 \\ 0 & \qquad\quad < \end{cases}$$

Randomization is not necessary here.

All that remains is to specify v. Equivalently, since σ^2 and $\|s_1\|^2$ are known, we can also just specify $v' = \sigma^2 \ln v + \frac{1}{2}||s_1||^2$.

Statistics of the Decision Variable $z = s_1^\top y$

To specify v, we need to know the mean and variance of $Z = s_1^\top Y$, under the assumption that the state is x_0, so that we can set the decision threshold to satisfy the false positive probability constraint.

When the state is x_0, we have $Y = W$ with $W \sim \mathcal{N}(0, \sigma^2 I)$. Clearly $Z = s^\top Y = s^\top W$ is Gaussian distributed. We can calculate the mean and variance of Z under the assumption that the state is x_0 as

$$\begin{aligned}
\mu_0 &:= \mathrm{E}[Z \,|\, x = s_0] = \mathrm{E}[s_1^\top Y \,|\, x = s_j] = s_1^\top \mathrm{E}[Y \,|\, x = s_0] = 0 \\
\sigma_0^2 &:= \mathrm{E}[(Z - s_1^\top s_0)^2 \,|\, x = s_0] = \mathrm{E}[Z Z^\top \,|\, x = s_0] = s_1^\top \mathrm{E}[W W^\top] s_1 \\
&= \sigma^2 \|s_1\|^2
\end{aligned}$$

with $\|s_1\|^2 = A^2 \sum_{k=0}^{n-1} r^{2k}$. A false positive occurs if the state is x_0 and $Z \geq v'$. Given the state is x_0, the decision variable $Z \sim \mathcal{N}(0, \sigma^2 \|s_1\|^2)$. Hence

$$P_{\mathsf{fp}} = Q\left(\frac{v'}{\sigma \|s_1\|}\right) \leq \alpha$$

Neyman-Pearson Decision Rule

Since

$$P_{\mathsf{fp}} = Q\left(\frac{v'}{\sigma\|s_1\|}\right) \leq \alpha$$

we can set this equal to α to get $v' = \sigma\|s_1\|Q^{-1}(\alpha)$. This decision threshold satisfies the false positive probability constraint with equality.

The N-P decision rule then

$$\rho^{\mathrm{NP}}(y) \quad = \quad \begin{cases} 1 & s_1^\top y \quad \geq \sigma\|s_1\|Q^{-1}(\alpha) \\ 0 & \qquad\quad < \end{cases}$$

and the probability of detection is

$$P_D = Q\left(\frac{\sigma\|s_1\|Q^{-1}(\alpha) - \|s_1\|^2}{\sigma\|s_1\|}\right) = Q\left(Q^{-1}(\alpha) - \frac{\|s_1\|}{\sigma}\right)$$

Asymptotics

Explain what happens as $n \to \infty$ for $0 < r < 1$, $r = 1$, and $r > 1$.

In general, we have $\|s_1\|^2 = A^2 \sum_{k=0}^{n-1} r^{2k}$. For the cases when $r = 1$ and $r > 1$, this clearly blows up as $n \to \infty$. Since

$$P_D = Q\left(Q^{-1}(\alpha) - \frac{\|s_1\|}{\sigma}\right)$$

the argument to the Q function is going to $-\infty$, hence $P_D \to 1$.

For $0 < r < 1$, we have $\|s_1\|^2 = A^2 \sum_{k=0}^{n-1} r^{2k} \to \frac{A^2}{1-r^2}$. Hence

$$P_D \to Q\left(Q^{-1}(\alpha) - \frac{A}{\sigma\sqrt{1-r^2}}\right)$$

The probability of detection improves with larger A, smaller σ, and/or r closer to 1.