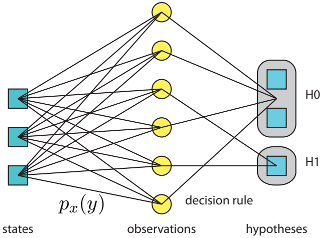
ECE531 Screencast 11.1: Introduction to Composite Hypothesis Testing

D. Richard Brown III

Worcester Polytechnic Institute

Introduction

Recall that **simple** hypothesis testing problems have one state per hypothesis. **Composite** hypothesis testing problems have at least one hypothesis containing more than one state.



Example: Y is Gaussian distributed with known variance σ^2 but unknown mean $\mu \in \mathbb{R}$. Given an observation $Y = y$, we want to decide if $\mu > \mu_0$.

- Only two hypotheses here: $\mathcal{H}_0 : \mu \leq \mu_0$ and $\mathcal{H}_1 : \mu > \mu_0$.
- This is a **binary composite** hypothesis testing problem with an uncountably infinite number of states.

Mathematical Model and Notation

- Let $\mathcal{X}$ be the set of states. $\mathcal{X}$ can now be
  - finite, e.g. $\{0, 1\}$,
  - countably infinite, e.g. $\{0, 1, \dots\}$, or
  - uncountably infinite, e.g. $[0, 1]$ or $\mathbb{R}^2$.
- We still assume a finite number of hypotheses $\mathcal{H}_0, \dots, \mathcal{H}_{M-1}$ with $M < \infty$. As before, hypotheses are defined as a partition on $\mathcal{X}$.
  - Composite HT: At least one hypothesis contains more than one state.
  - If $\mathcal{X}$ is infinite (countably or uncountably), at least one hypothesis must contain an infinite number of states.
- Let $C^\top(x) = [C_{0,x}, \dots, C_{M-1,x}]^\top$ be the vector of costs associated with making decision $i \in \mathcal{Z} = \{0, \dots, M-1\}$ when the state is x.
- Let $\mathcal{Y}$ be the set of observations.
- Each state $x \in \mathcal{X}$ has an associated conditional pmf or pdf $p_x(y)$ that specifies how states are mapped to observations. These conditional pmfs/pdfs are, as always, assumed to be known.

The Uniform Cost Assignment in Composite HT Problems

In general, for $i = 0, \dots, M-1$ and $x \in \mathcal{X}$, the UCA is given as

$$C_i(x) = \begin{cases} 0 & \text{if } x \in \mathcal{H}_i \\ 1 & \text{otherwise.} \end{cases}$$

We can also form a cost vector $C(x) = [C_0(x), \dots, C_{M-1}(x)]^\top$.

Example: A random variable Y is Gaussian distributed with known variance σ^2 but unknown mean $\mu \in \mathbb{R}$. Given an observation $Y = y$, we want to decide among $\mathcal{H}_0 : \mu < -1$, $\mathcal{H}_1 : -1 \leq \mu \leq 1$, and $\mathcal{H}_2 : \mu > 1$.

Uniform cost assignment ($x = \mu$):

$$C(\mu) = \begin{cases} [0,1,1]^\top & \text{if } \mu < -1 \\ [1,0,1]^\top & \text{if } -1 \leq \mu \leq 1 \\ [1,1,0]^\top & \text{if } \mu > 1 \end{cases}$$

Other cost assignments, e.g. squared error, are also possible.

Conditional Risk

Given decision rule ρ, the conditional risk for state $x \in \mathcal{X}$ can be written as

$$\begin{aligned} R_x(\rho) &= \int_{\mathcal{Y}} \rho^\top(y) C(x) p_x(y)\, dy \\ &= \mathrm{E}[\rho^\top(Y) C(x) \,|\, \text{state is } x] \\ &= \mathrm{E}_x[\rho^\top(Y) C(x)] \end{aligned}$$

where the last expression is a common shorthand notation used in many textbooks.

Recall that the decision rule

$$\rho^\top(y) = [\rho_0(y), \dots, \rho_{M-1}(y)]$$

where $0 \leq \rho_i(y) \leq 1$ specifies the probability of deciding $\mathcal{H}_i$ when you observe $Y = y$. Also recall that $\sum_i \rho_i(y) = 1$ for all $y \in \mathcal{Y}$.