ECE531 Screencast 11.3: Bayesian Composite Hypothesis Testing Example

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Problem Setup

Suppose $\mathcal{X} = [0,3)$ and we want to decide between three hypotheses

$$\mathcal{H}_0 : 0 \le x < 1 \qquad \mathcal{H}_1 : 1 \le x < 2 \qquad \mathcal{H}_2 : 2 \le x < 3$$

We get one observation $Y = x + \eta$ where η is uniformly distributed on $[-1,1]$. From this setup, the conditional distribution of observation Y is

$$p_x(y) = \begin{cases} \frac{1}{2} & x-1 \le y \le x+1 \\ 0 & \text{otherwise} \end{cases}$$

Further assume the UCA and a uniform prior on the states. This implies

$$C(x) = \begin{cases} [0,1,1]^\top & 0 \le x < 1 \\ [1,0,1]^\top & 1 \le x < 2 \\ [1,1,0]^\top & 2 \le x < 3 \end{cases}$$

and

$$\pi(x) = \begin{cases} \frac{1}{3} & 0 \le x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Commodity Costs (1 of 3)

We want to find the index corresponding to the minimum ‘commodity cost”:

$$\begin{aligned} g_0(y,\pi) &= \frac{1}{3}\int_1^3 p_x(y)\,dx \\ g_1(y,\pi) &= \frac{1}{3}\int_0^1 p_x(y)\,dx + \frac{1}{3}\int_2^3 p_x(y)\,dx \\ g_2(y,\pi) &= \frac{1}{3}\int_0^2 p_x(y)\,dx \end{aligned}$$

We can simplify this a bit by defining

$$w(y) \;:=\; \frac{1}{3}\int_0^3 p_x(y)\,dx.$$

Note that $w(y)$ is not a function of x or i, i.e. given the observation y, $w(y)$ is just a constant.

Further defining $f_i(y,\pi) := w(y) - g_i(y,\pi)$ for $i \in \{0,1,2\}$, we can write

$$\arg\min_{i\in\{0,1,2\}} g_i(y,\pi) \quad\Leftrightarrow\quad \arg\max_{i\in\{0,1,2\}} f_i(y,\pi).$$

Commodity Costs (2 of 3)

We can explicitly write out the $f_i(y,\pi)$ terms as

$$\begin{aligned}
f_0(y,\pi) &= \frac{1}{3}\int_0^3 p_x(y) - \frac{1}{3}\int_1^3 p_x(y)\,dx = \frac{1}{3}\int_0^1 p_x(y)\,dx \\
f_1(y,\pi) &= \frac{1}{3}\int_0^3 p_x(y) - \frac{1}{3}\left(\int_0^1 p_y(x)\,dx + \int_2^3 p_y(x)\,dx\right) = \frac{1}{3}\int_1^2 p_x(y)\,dx \\
f_2(y,\pi) &= \frac{1}{3}\int_0^3 p_x(y) - \frac{1}{3}\int_0^2 p_y(x)\,dx = \frac{1}{3}\int_2^3 p_x(y)\,dx.
\end{aligned}$$

Note that the $\frac{1}{3}$ factor common to each $f_i(y,\pi)$ is irrelevant to the maximization. Hence, defining $h_i(y,\pi) := 3f_i(y,\pi)$ for $i \in \{0,1,2\}$, we can say that

$$\arg\min_{i\in\{0,1,2\}} g_i(y,\pi) \quad\Leftrightarrow\quad \arg\max_{i\in\{0,1,2\}} f_i(y,\pi) \quad\Leftrightarrow\quad \arg\max_{i\in\{0,1,2\}} h_i(y,\pi).$$

with

$$h_0(y,\pi) = \int_0^1 p_x(y)\,dx \qquad h_1(y,\pi) = \int_1^2 p_x(y)\,dx \qquad h_2(y,\pi) = \int_2^3 p_x(y)\,dx.$$

Commodity Costs (3 of 3)

In order to evaluate these integrals, we should write $p_x(y)$ as an explicit function of x, thinking of y as a constant.

As an example, suppose we get the observation $y = 0$. This observation tells us that $-1 \leq x \leq 1$.

We can write

$$p_x(y) = \begin{cases} \frac{1}{2} & y - 1 \leq x \leq y + 1 \\ 0 & \text{otherwise.} \end{cases}$$

You can confirm that this expression is identical to the earlier one on slide 2.

Bayesian Decision Rule (1 of 2)

The Bayesian decision rule selects

$$\delta^{B\pi} = \arg \min_{i\in\{0,1,2\}} g_i(y,\pi) = \arg \max_{i\in\{0,1,2\}} h_i(y,\pi)$$

with

$$h_0(y,\pi) = \int_0^1 p_x(y)\,dx \qquad h_1(y,\pi) = \int_1^2 p_x(y)\,dx \qquad h_2(y,\pi) = \int_2^3 p_x(y)\,dx$$

and

$$p_x(y) \quad = \quad \begin{cases} \frac{1}{2} & y-1 \le x \le y+1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $y = 0.5$. Then $p_x(y)$ is nonzero only on $x \in [-0.5, 1.5]$ and we can write

$$h_0(y,\pi) = \int_0^1 0.5\,dx = 0.5$$

$$h_1(y,\pi) = \int_1^2 p_x(y)\,dx = \int_1^{1.5} 0.5\,dx + \int_{1.5}^2 0\,dx = 0.25$$

$$h_2(y,\pi) = \int_2^3 p_x(y)\,dx = \int_2^3 0\,dx = 0$$

and we should select $\mathcal{H}_0$.

Bayesian Decision Rule (2 of 2)

Listing the relevant regions of y, we can write

$$y \in (-1,0) : h_0 > 0,\ h_1 = h_2 = 0 \quad \Rightarrow \text{decide } \mathcal{H}_0$$
$$y \in (0,1) : h_0 = 0.5,\ 0 < h_1 < 0.5,\ h_2 = 0 \quad \Rightarrow \text{decide } \mathcal{H}_0$$
$$y \in (1,2) : 0 < h_0 < 0.5,\ h_1 = 0.5,\ 0 < h_2 < 0.5 \quad \Rightarrow \text{decide } \mathcal{H}_1$$
$$y \in (2,3) : h_0 = 0,\ 0 < h_1 < 0.5,\ h_2 = 0.5 \quad \Rightarrow \text{decide } \mathcal{H}_2$$
$$y \in (3,4) : h_0 = h_1 = 0,\ h_2 > 0 \quad \Rightarrow \text{decide } \mathcal{H}_2$$

Hence,

$$\delta^{B\pi}(y) = \begin{cases} 0 & y \le 1 \\ 1 & 1 < y < 2 \\ 2 & y \ge 2 \end{cases}$$

where

$$\mathcal{H}_0 : 0 \le x < 1 \qquad \mathcal{H}_1 : 1 \le x < 2 \qquad \mathcal{H}_2 : 2 \le x < 3.$$