ECE531 Screencast 11.5: Uniformly Most Powerful Decision Rules

D. Richard Brown III

Worcester Polytechnic Institute

Monotone Likelihood Ratio

Despite UMP decision rules existing only in "special cases", lots of real problems fall into these special cases. One such class of problems is the class of binary composite HT problems with monotone likelihood ratios.

Definition (Lehmann, Testing Statistical Hypotheses, p.78)

The real-parameter family of densities $p_x(y)$ for $x \in \mathcal{X} \subset \mathbb{R}$ is said to have **monotone likelihood ratio** if there exists a real-valued function $T(y)$ that only depends on y such that, for any $x_1 \in \mathcal{X}$ and $x_0 \in \mathcal{X}$ with $x_0 < x_1$, the distributions $p_{x_0}(y)$ and $p_{x_1}(y)$ are distinct and the ratio

$$L_{x_1/x_0}(y) := \frac{p_{x_1}(y)}{p_{x_0}(y)} = f(x_0, x_1, T(y))$$

is a non-decreasing function of $T(y)$.

Monotone Likelihood Ratio: Example

Consider the Laplacian density $p_x(y) = \frac{b}{2}e^{-b|y-x|}$ with $b > 0$ and $\mathcal{X} = [0, \infty)$.
Let's see if this family of densities has monotone likelihood ratio...

$$L_{x_1/x_0}(y) := \frac{p_{x_1}(y)}{p_{x_0}(y)} = e^{b(|y-x_0|-|y-x_1|)}$$

Since $b > 0$, $L_{x_1/x_0}(y)$ will be non-decreasing in $T(y) = y$ provided that $|y - x_0| - |y - x_1|$ is non-decreasing in y for all $0 \le x_0 < x_1$. We can write

$$|y - x_0| - |y - x_1| = \begin{cases} x_0 - x_1 & y < x_0 \\ 2y - x_1 - x_0 & x_0 \le y \le x_1 \\ x_1 - x_0 & y > x_1. \end{cases}$$

Note that $x_0 - x_1 < 0$ is a constant and $x_1 - x_0 > 0$ is also a constant with respect to y. Hence we can see that the Laplacian family of densities $p_x(y) = \frac{b}{2}e^{-b|y-x|}$ with $b > 0$ and $\mathcal{X} = [0, \infty)$ is monotone in $T(y) = y$.

Existence of a UMP Decision Rule for Monotone LR

Theorem (Lehmann, Testing Statistical Hypotheses, p.78)

Let $\mathcal{X}$ be a subinterval of the real line. Fix $\lambda \in \mathcal{X}$ and define the hypotheses $\mathcal{H}_0 : x \in \mathcal{X}_0 = \{x \leq \lambda\}$ versus $\mathcal{H}_1 : x \in \mathcal{X}_1 = \{x > \lambda\}$. If the family of densities $p_x(y)$ are distinct for all $x \in \mathcal{X}$ and has monotone likelihood ratio in $T(y)$, then the decision rule

$$\rho(y) = \begin{cases} 1 & T(y) > v \\ \gamma & T(y) = v \\ 0 & T(y) < v \end{cases}$$

where v and γ are selected so that $P_{\textsf{fp},x=\lambda} = \alpha$ is UMP for testing $\mathcal{H}_0 : x \in \mathcal{X}_0$ versus $\mathcal{H}_1 : x \in \mathcal{X}_1$.

Coin Flipping Example

Suppose we have a coin with $\text{Prob}(\text{heads}) = x$, where $0 \le x \le 1$ is the unknown state. We flip the coin n times and observe the number of heads $Y = y \in \{0, \dots, n\}$. We know that, conditioned on the state x, the observation Y is distributed as

$$p_x(y) = \binom{n}{y} x^y (1-x)^{n-y}.$$

Fix $0 < \lambda < 1$ and define our composite binary hypotheses as

$$\begin{aligned} \mathcal{H}_0 &: Y \sim p_x(y) \text{ for } x \in [0, \lambda] = \mathcal{X}_0 \\ \mathcal{H}_1 &: Y \sim p_x(y) \text{ for } x \in (\lambda, 1] = \mathcal{X}_1 \end{aligned}$$

Does there exist a UMP decision rule with significance level α?

- The $p_x(y)$ are distinct for all $x \in \mathcal{X}$.
- This family of densities has a monotone likelihood ratio for $T(y) = y$ (check this).

Coin Flipping Example

According the the theorem, a UMP decision rule will be

$$\rho(y) = \begin{cases} 1 & y > v \\ \gamma & y = v \\ 0 & y < v \end{cases}$$

with v and γ selected so that $P_{\mathsf{fp},x=\lambda} = \alpha$. We just have to find v and γ.

Coin Flipping Example

To find v, we can use our procedure from **simple** binary N-P hypothesis testing. We want to find the smallest value of v such that

$$P_{\mathsf{fp},x=\lambda}(\delta^v) = \sum_{j>v} \underbrace{\mathrm{Prob}[y=j \,|\, x=\lambda]}_{p_\lambda(y)} = \sum_{j>v}^{n} \binom{n}{j} \lambda^j (1-\lambda)^{n-j} \leq \alpha.$$

Once we find the appropriate value of v, if $P_{\mathsf{fp},x=\lambda}(\delta^v) = \alpha$, then we are done. We can set γ to any arbitrary value in $[0,1]$ in this case.
If $P_{\mathsf{fp},x=\lambda}(\delta^v) < \alpha$, then we have to find γ such that $P_{\mathsf{fp},x=\lambda}(\rho) = \alpha$. The UMP decision rule is just the usual randomization

$$\rho \;=\; (1-\gamma)\delta^v + \gamma\delta^{v-\epsilon}$$

and, in this case,

$$\gamma = \frac{\alpha - \sum_{j=v+1}^{n} \binom{n}{j} \lambda^j (1-\lambda)^{n-j}}{\binom{n}{v} \lambda^v (1-\lambda)^{n-v}}.$$

Power Function

For binary composite hypothesis testing problems with $\mathcal{X}$ a subinterval of the real line and a particular decision rule ρ, we can define the power function of ρ as

$$\beta(x) \quad := \quad \mathrm{Prob}(\rho \text{ decides } 1 \,|\, \text{state is } x)$$

- For each $x \in \mathcal{X}_1$, $\beta(x)$ is the probability of a true positive (the probability of detection).
- For each $x \in \mathcal{X}_0$, $\beta(x)$ is the probability of a false positive.
- Hence, a plot of $\beta(x)$ versus x displays both the probability of a false positive and the probability of detection of ρ for all states $x \in \mathcal{X}$.

Our UMP decision rule for the coin flipping problem specifies that we always decide 1 when we observe $v+1$ or more heads, and we decide 1 with probability γ if we observe v heads. Hence,

$$\beta(x) \quad = \quad \sum_{j=v+1}^{n} \binom{n}{j} x^j (1-x)^{n-j} + \gamma \binom{n}{v} x^v (1-x)^{n-v}$$

Coin Flipping Example: Power Function $\lambda = 0.5$, $\alpha = 0.1$