ECE531 Screencast 11.7: Generalized Likelihood Ratio Test

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The Generalized Likelihood Ratio Test

We focus here on a binary composite hypothesis testing problem with $\mathcal{H}_0 : x \in \mathcal{X}_0$ versus $\mathcal{H}_1 : x \in \mathcal{X} \backslash \mathcal{X}_0$.

The main idea of the GLRT is to

- get an observation y
- estimate the most likely value of x under $\mathcal{H}_0$ (call this $\hat{x}_0$)
- estimate the most likely value of x under $\mathcal{H}_1$ (call this $\hat{x}_1$)

and then use those estimates as "truth" so that we have a simple binary hypothesis testing problem $\mathcal{H}_0 : x = \hat{x}_0$ versus $\mathcal{H}_1 : x = \hat{x}_1$.

You can then specify the decision rule via the standard N-P lemma for simple binary hypothesis testing.

Connection to Bayesian Composite Hypothesis Testing

Let $p_i(y;x)$ be the family of densities parameterized by x under hypothesis $\mathcal{H}_i$. Often we have $p_0(y;x) = p_1(y;x)$, but these densities don't have to have the same form.

With the GLRT, we decide $\mathcal{H}_1$ if

$$\frac{\max_{x \in \mathcal{X} \setminus \mathcal{X}_0} p_1(y;x)}{\max_{x \in \mathcal{X}_0} p_0(y;x)} > v$$

In the case of Bayesian binary hypothesis testing, we can show that we decide $\mathcal{H}_1$ if

$$\frac{\int_{x \in \mathcal{X} \setminus \mathcal{X}_0} p_1(y|x)\, dx}{\int_{x \in \mathcal{X}_0} p_0(y|x)\, dx} > v$$

Intuition: The GLRT decision rule compares the <u>most likely</u> model in $\mathcal{H}_1$ to the <u>most likely</u> model in $\mathcal{H}_0$. The Bayesian decision rule compares the <u>average</u> model in $\mathcal{H}_1$ to the <u>average</u> model in $\mathcal{H}_0$, using the prior probability distribution on the unknown state.

GLRT Example (part 1 of 4)

Suppose we get a vector observation $Y \sim \mathcal{N}(Hx, \sigma^2 I)$ in $\mathbb{R}^n$ with σ^2 known and have two hypotheses: $\mathcal{H}_0 : x = 0$ versus $\mathcal{H}_1 : x \neq 0$ for $x \in \mathbb{R}^N$. We assume $H^\top H$ is invertible.

Given $Y = y$, we want to find the most likely x under $\mathcal{H}_0$ and $\mathcal{H}_1$.

For the denominator of the GLRT, we compute $\max_{x \in \mathcal{X}_0} p_0(y; x)$. But $\mathcal{X}_0 = \{0\}$, so the maximization is trivial. The denominator of the GLRT is

$$\max_{x \in \mathcal{X}_0} p_0(y; x) = p_0(y; x = 0) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{y^\top y}{2\sigma^2}\right)$$

For the numerator, we compute $\max_{x \neq 0} p_1(y; x)$. Since this is a linear Gaussian model, we can use the known results for the MLE to write $\hat{x}_1 = (H^\top H)^{-1} H^\top y$. Hence

$$\max_{x \in \mathcal{X} \setminus \mathcal{X}_0} p_1(y; x) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{(y - H\hat{x}_1)^\top (y - H\hat{x}_1)}{2\sigma^2}\right)$$

GLRT Example (part 2 of 4)

The GLRT is then

$$\frac{\max_{x \in \mathcal{X} \setminus \mathcal{X}_0} p_1(y;x)}{\max_{x \in \mathcal{X}_0} p_0(y;x)} = \frac{\exp\left(-\frac{(y-H\hat{x}_1)^\top (y-H\hat{x}_1)}{2\sigma^2}\right)}{\exp\left(-\frac{y^\top y}{2\sigma^2}\right)} > v$$

with $\hat{x}_1 = (H^\top H)^{-1} H^\top y = Py$. Simplifying and taking the log of both sides, we have

$$\begin{aligned}
\frac{-1}{2\sigma^2}\left(y^\top y - 2y^\top H\hat{x}_1 + \hat{x}_1^\top H^\top H \hat{x}_1 - y^\top y\right) &> v' \\
\Leftrightarrow \quad 2y^\top H(H^\top H)^{-1} H^\top y - y^\top H (H^\top H)^{-1} H^\top H (H^\top H)^{-1} H^\top y &> v'' \\
\Leftrightarrow \quad y^\top H(H^\top H)^{-1} H^\top y &> v'' \\
\Leftrightarrow \quad y^\top P y &> v''
\end{aligned}$$

where we choose v'' to satisfy the false positive probability constraint.

GLRT Example (part 3 of 4)

Note that we can write $H = QR$ where $Q \in \mathbb{R}^{N \times n}$ is a matrix with orthonormal columns and $R \in \mathbb{R}^{n \times n}$ is an invertible upper triangular matrix. This is called the (reduced) QR factorization.

Then

$$\begin{aligned} P &= H(H^\top H)^{-1} H^\top \\ &= QR(R^\top Q^\top QR)^{-1} R^\top Q^\top \\ &= QR(R^\top I R)^{-1} R^\top Q^\top \\ &= QRR^{-1}(R^\top)^{-1} R^\top Q^\top \\ &= QQ^\top \end{aligned}$$

Hence our decision statistic is

$$Y^\top P Y = Y^\top Q Q^\top Y = Z^\top Z.$$

What is the distribution of Z under $\mathcal{H}_0$?

GLRT Example (part 4 of 4)

We have $Z = Q^\top Y$ with $Y \sim \mathcal{N}(0, \sigma^2 I)$ under $\mathcal{H}_0$.

Clearly Z is Gaussian with $\mathrm{E}[Z] = E[Q^\top Y] = 0$.

We can also compute

$$\mathrm{E}[ZZ^\top] = E[Q^\top Y Y^\top Q] = Q^\top (\sigma^2 I_{N\times N}) Q = \sigma^2 I_{n\times n}$$

So $Z \sim \mathcal{N}(0, \sigma^2 I)$ in $\mathbb{R}^n$ and $\frac{Z^\top Z}{\sigma^2} \sim \chi^2_n$.

Given a false positive probability constraint α, you can use the inverse CDF of the Chi-squared distribution with n degrees of freedom to find the optimum decision threshold.

For example, set $\alpha = 0.01$ and $n = 10$. In Matlab, you can use `v = chi2inv(0.99,10)` to get $v = 23.2093$. Then we decide $\mathcal{H}_1$ if $Z^\top Z > v\sigma^2$.