ECE531 Screencast 12.5: Detection of a Known Signal with Unknown Parameters in a Linear Model

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Linear Model

Suppose we have observations given by

$$y = Hx + w$$

where $H \in \mathbb{R}n \times p$ is known, $x \in \mathbb{R}^{p \times 1}$ is a vector of parameters/states, at least some of which are unknown, and $W \in \mathbb{R}^{n \times 1}$ with $W \sim \mathcal{N}(0, \sigma^2 I)$ is AWGN with σ^2 known.

We want to decide between

$$\begin{aligned} \mathcal{H}_0 &: x = 0 \\ \mathcal{H}_1 &: x \neq 0 \end{aligned}$$

Many problems fit into this linear model, even some that aren't obvious.

Example: Sinusoidal Signal with Unknown Amp. and Phase

Suppose the signal we are trying to detect is

$$s_k = a\cos(\omega k + \phi)$$

for $k = 0, \dots, n-1$ with unknown a and ϕ. We can write

$$s_k = \alpha_1 \cos(\omega k) + \alpha_2 \sin(\omega k)$$

with $\alpha_1^2 + \alpha_2^2 = a^2$. Letting the unknown parameters $x = [\alpha_1, \alpha_2]^\top$, we have

$$y = Hx + w$$

with

$$H = \begin{bmatrix} 1 & 0 \\ \cos(\omega_0) & \sin(\omega_0) \\ \vdots & \vdots \\ \cos(\omega_0(n-1)) & \sin(\omega_0(n-1)) \end{bmatrix}$$

GLRT for Linear Model

Since the MLE for the linear model with AWGN is simply

$$\hat{x} = (H^\top H)^{-1} H^\top y$$

the GLRT decision rule can be written as follows. We decide $\mathcal{H}_1$ if

$$\begin{aligned}
& \frac{p_1(y;\hat{x})}{p_0(y)} > v \\
\Leftrightarrow \quad & \frac{\exp\left(-\frac{(y-H\hat{x})^\top(y-H\hat{x})}{2\sigma^2}\right)}{\exp\left(-\frac{y^\top y}{2\sigma^2}\right)} > v \\
\Leftrightarrow \quad & 2\hat{x}^\top H^\top y - \hat{x} H^\top H \hat{x} > v' \\
\Leftrightarrow \quad & 2y^\top H(H^\top H)^{-1} H^\top y - y^\top H(H^\top H)^{-1} H^\top H (H^\top H)^{-1} H^\top y > v' \\
\Leftrightarrow \quad & y^\top H(H^\top H)^{-1} H^\top y > v'
\end{aligned}$$

with v' chosen to satisfy $P_{\mathsf{fp}} \le \alpha$. The decision statistic is χ^2 distributed, so we can get exact expressions for P_{fp} and P_D.

Bayesian Detection for Linear Model with Gaussian Prior

In the Bayesian case with $Y = Hx + W$, we assume a Gaussian prior on x such that

$$\pi(x) = \pi_0 \delta(x) + (1-\pi_0)\frac{1}{(2\pi)^{n/2}|\Sigma_x|^{1/2}} \exp\left(-\frac{(x-\mu_x)^\top \Sigma_x^{-1}(x-\mu_x)}{2}\right).$$

If we let $s = Hx$ then s is also Gaussian with $\mu_s = H\mu_x$ and $\Sigma_s = H\Sigma_x H^\top$. We assume μ_s and Σ_s are known and s is independent of the noise $W \sim \mathcal{N}(0, \sigma^2 I)$.

We want to decide between

$$\begin{aligned}\mathcal{H}_0 &: x = 0 \;\Leftrightarrow\; s = 0\\ \mathcal{H}_1 &: x \neq 0 \;\Leftrightarrow\; s \neq 0\end{aligned}$$

which is equivalent to

$$\begin{aligned}\mathcal{H}_0 &: Y \sim \mathcal{N}(0, \sigma^2 I)\\ \mathcal{H}_1 &: Y \sim \mathcal{N}(\mu_s, \Sigma_s + \sigma^2 I)\end{aligned}$$

This is actually a simple binary hypothesis testing problem!

Bayesian Detection for Linear Model with Gaussian Prior

Since this problem is simple and binary, we know we decide $\mathcal{H}_1$ when

$$
\begin{aligned}
& \frac{p_1(y)}{p_0(y)} > \frac{\pi_0}{1-\pi_0} \\
\Leftrightarrow \quad & \frac{\frac{1}{(2\pi)^{n/2}|\Sigma_s+\sigma^2 I|^{1/2}} \exp\left(-\frac{(y-\mu_s)^\top(\Sigma_s+\sigma^2 I)^{-1}(y-\mu_s)}{2}\right)}{\frac{1}{(2\pi)^{n/2}|\sigma^2 I|^{1/2}} \exp\left(-\frac{y^\top y}{2\sigma^2}\right)} > \frac{\pi_0}{1-\pi_0} \\
\Leftrightarrow \quad & \frac{|\sigma^2 I|^{1/2}}{|\Sigma_s+\sigma^2 I|^{1/2}} \exp\left(-\frac{(y-\mu_s)^\top(\Sigma_s+\sigma^2 I)^{-1}(y-\mu_s)}{2} + \frac{y^\top y}{2\sigma^2}\right) > \frac{\pi_0}{1-\pi_0} \\
\Leftrightarrow \quad & \frac{y^\top y}{\sigma^2} - (y-\mu_s)^\top(\Sigma_s+\sigma^2 I)^{-1}(y-\mu_s) > 2\ln\left(\frac{\pi_0|\Sigma_s+\sigma^2 I|^{1/2}}{(1-\pi_0)|\sigma^2 I|^{1/2}}\right) \\
\Leftrightarrow \quad & y^\top A y + \mu_s^\top B y > 2\ln\left(\frac{\pi_0|\Sigma_s+\sigma^2 I|^{1/2}}{(1-\pi_0)|\sigma^2 I|^{1/2}}\right) + \mu_s^\top(\Sigma_s+\sigma^2 I)^{-1}\mu_s
\end{aligned}
$$

with

$$
A = \frac{1}{\sigma^2} I - (\Sigma_s + \sigma^2 I)^{-1} \qquad \text{and} \qquad B = 2(\Sigma_s+\sigma^2 I)^{-1}
$$