ECE531 Screencast 12.6: Kay vII Problems 7.1 and 7.2

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Problem Setup

We have a finite state space $\mathcal{X} = \{-1, 0, +1\}$ and get observations

$$Y_k = x + W_k$$

for $k = 0, \dots, n-1$ with $W \sim \mathcal{N}(0, \sigma^2 I)$. The hypotheses are

$$\mathcal{H}_0 : x \in \mathcal{X}_0 = \{0\}$$
$$\mathcal{H}_1 : x \in \mathcal{X}_1 = \{-1, 1\}$$

In problem 7.1, we are first asked if a UMP decision rule exists. If not, then we are asked to find the GLRT.

To check if a UMP decision rule exists, we just need to see if the critical regions are the same for the simple binary hypothesis tests:

1. $\mathcal{H}_0 : x = 0$ vs $\mathcal{H}_1' : x = 1$
2. $\mathcal{H}_0 : x = 0$ vs $\mathcal{H}_1'' : x = -1$

UMP Decision Rule

For $\mathcal{H}_0 : x = 0$ vs $\mathcal{H}_1' : x = 1$, we decide $\mathcal{H}_1'$ when

$$\frac{p_1(y)}{p_0(y)} = \frac{\exp\left(-\frac{(y-1)^\top(y-1)}{2\sigma^2}\right)}{\exp\left(-\frac{y^\top y}{2\sigma^2}\right)} = \exp\left(\frac{2 \cdot 1^\top y - n}{2\sigma^2}\right) > v$$

Hence the decision rule will be of the form $1^\top y > v'$.

For $\mathcal{H}_0 : x = 0$ vs $\mathcal{H}_1'' : x = -1$, we decide $\mathcal{H}_1''$ when

$$\frac{p_1(y)}{p_0(y)} = \frac{\exp\left(-\frac{(y+1)^\top(y+1)}{2\sigma^2}\right)}{\exp\left(-\frac{y^\top y}{2\sigma^2}\right)} = \exp\left(\frac{-2 \cdot 1^\top y - n}{2\sigma^2}\right) > v$$

Hence the decision rule will be of the form $1^\top y < v'$. The critical region is changing, so a UMP decision rule doesn't exist.

GLRT Decision Rule (part 1 of 2)

To compute the GLRT decision rule, we need to compute the MLE of x under $\mathcal{H}_0$ and $\mathcal{H}_1$. Under the simple hypothesis $\mathcal{H}_0$ there is nothing to do.

Under the composite hypothesis $\mathcal{H}_1$, the MLE is just the value of $x \in \{-1,+1\}$ that makes the observation y more likely. So, given y, we choose

$$\hat{x} = \arg \max_{x \in \{-1,+1\}} \exp\left(-\frac{(y-1x)^\top (y-1x)}{2\sigma^2}\right)$$

which is equivalent to

$$\begin{aligned}\hat{x} &= \arg \min_{x \in \{-1,+1\}} (y-1x)^\top (y-1x) \\ &= \arg \min_{x \in \{-1,+1\}} y^\top y - 2x1^\top y + 1^\top 1 x^2 \\ &= \arg \max_{x \in \{-1,+1\}} x1^\top y\end{aligned}$$

which is equivalent to $\hat{x} = \mathrm{sign}\left(1^\top y\right)$.

GLRT Decision Rule (part 2 of 2)

So the GLRT decision rule decides $\mathcal{H}_1$ if

$$\begin{aligned} & \frac{p_1(y;\hat{x})}{p_0(y)} > v \\ \Leftrightarrow \quad & \frac{\exp\left(-\frac{(y-1\hat{x})^\top(y-1\hat{x})}{2\sigma^2}\right)}{\exp\left(-\frac{y^\top y}{2\sigma^2}\right)} > v \\ \Leftrightarrow \quad & 2\hat{x}1^\top y - n > v' \\ \Leftrightarrow \quad & |1^\top y| > v'' \end{aligned}$$

with v'' chosen to satisfy the false positive probability constraint.

Bayesian Decision Rule (part 1 of 3)

In Problem 7.2 we are asked to find a Bayesian decision rule (assume the UCA). The prior probabilities are given as

$$\begin{aligned}\pi_0 &= \mathrm{Prob}(x=0) = \pi_0 \\ \pi_1 &= \mathrm{Prob}(x=-1) = \frac{1-\pi_0}{2} \\ \pi_2 &= \mathrm{Prob}(x=+1) = \frac{1-\pi_0}{2}\end{aligned}$$

Since we have a finite number of states and only two hypotheses, our Bayesian decision rule will be of the form

$$\delta^{B\pi}(y) = \begin{cases} 1 & \frac{g_0(y,\pi)}{g_1(y,\pi)} > 1 \\ 0 & \text{otherwise.} \end{cases}$$

where

$$g_i(y,\pi) = \sum_{j=0}^{N-1} C_{i,j}\pi_j p_j(y)$$

is the usual "commodity cost" associated with hypothesis $\mathcal{H}_i$.

Bayesian Decision Rule (part 2 of 3)

Under the UCA, we have

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

hence

$$g_0(y,\pi) = \sum_{j=0}^{2} C_{0,j}\pi_j p_j(y) = \frac{1-\pi_0}{2}\left(p_1(y) + p_2(y)\right)$$

$$g_1(y,\pi) = \sum_{j=0}^{2} C_{1,j}\pi_j p_j(y) = \pi_0 p_0(y)$$

and

$$\frac{g_0(y,\pi)}{g_1(y,\pi)} = \left(\frac{1-\pi_0}{2\pi_0}\right)\frac{p_1(y)+p_2(y)}{p_0(y)}$$

Bayesian Decision Rule (part 3 of 3)

Hence we decide $\mathcal{H}_1$ when

$$
\begin{aligned}
& \left(\frac{1-\pi_0}{2\pi_0}\right)\frac{p_1(y)+p_2(y)}{p_0(y)} > 1 \\
\Leftrightarrow \quad & \frac{p_1(y)+p_2(y)}{p_0(y)} > \frac{2\pi_0}{1-\pi_0} \\
\Leftrightarrow \quad & \frac{\exp\left(-\frac{(y-1)^\top(y-1)}{2\sigma^2}\right)+\exp\left(-\frac{(y+1)^\top(y+1)}{2\sigma^2}\right)}{\exp\left(-\frac{y^\top y}{2\sigma^2}\right)} > \frac{2\pi_0}{1-\pi_0} \\
\Leftrightarrow \quad & \exp\left(\frac{1^\top y - n/2}{\sigma^2}\right)+\exp\left(\frac{-1^\top y - n/2}{\sigma^2}\right) > \frac{2\pi_0}{1-\pi_0} \\
\Leftrightarrow \quad & \exp\left(\frac{1^\top y}{\sigma^2}\right)+\exp\left(\frac{-1^\top y}{\sigma^2}\right) > \frac{2\pi_0\exp\left(\frac{n/2}{\sigma^2}\right)}{1-\pi_0} \\
\Leftrightarrow \quad & \left|1^\top y\right| > v
\end{aligned}
$$

which is the same form as the GLRT but v will be different.