ECE531 Screencast 13.4: Generalized Likelihood Ratio Test Detection with Unknown AWGN Noise Variance

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GLRT Decision Rules for Composite HT Problems

Given a composite hypothesis testing problem with unknown state x. Recall the main idea of the GLRT:

- get an observation y
- estimate the most likely value of x under $\mathcal{H}_0$ (call this $\hat{x}_0$)
- estimate the most likely value of x under $\mathcal{H}_1$ (call this $\hat{x}_1$)

and then form the GLRT

$$\frac{p_1(y;\hat{x}_1)}{p_0(y;\hat{x}_0)} > v$$

where v is selected to satisfy the false positive probability constraint.

GLRT: Known Signal in AWGN with Unknown Variance

We have the binary hypothesis testing problem

$$\begin{aligned}\mathcal{H}_0 &: Y = W \\ \mathcal{H}_1 &: Y = s + W\end{aligned}$$

with $s \in \mathbb{R}^n$ known and $W \sim \mathcal{N}(0, \sigma^2 I)$ with σ^2 unknown.

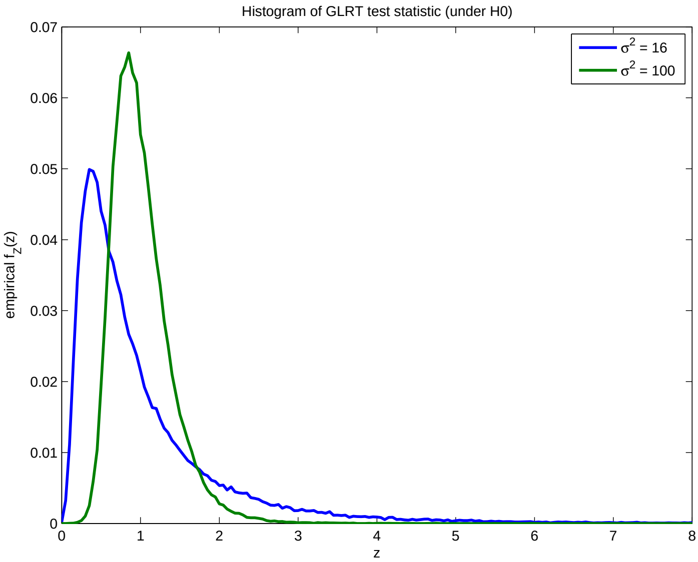
The MLE of σ^2 under $\mathcal{H}_0$ is $\hat{\sigma_0}^2 = \frac{1}{n} y^\top y$ and the MLE of σ^2 under $\mathcal{H}_1$ is $\hat{\sigma_1}^2 = \frac{1}{n}(y-s)^\top (y-s)$.

The GLRT (with $x = \sigma^2$) is then

$$\frac{\max_{x \in \mathcal{X} \setminus \mathcal{X}_0} p_1(y;x)}{\max_{x \in \mathcal{X}_0} p_0(y;x)} = \frac{\frac{1}{(2\pi\hat{\sigma_1}^2)^{n/2}} \exp\left(-\frac{(y-s)^\top (y-s)}{2\hat{\sigma_1}^2}\right)}{\frac{1}{(2\pi\hat{\sigma_0}^2)^{n/2}} \exp\left(-\frac{y^\top y}{2\hat{\sigma_0}^2}\right)} = \left(\frac{\hat{\sigma_0}^2}{\hat{\sigma_1}^2}\right)^{n/2} > v$$

Unfortunately, the statistics of $Z = \frac{\hat{\sigma_0}^2}{\hat{\sigma_1}^2}$ depend on σ^2 under $\mathcal{H}_0$, so the threshold v will depend on σ^2 to satisfy a false positive probability constraint. This detector is not CFAR.

GLRT: Known Signal in AWGN with Unknown Variance



GLRT for Classical Linear Model in AWGN with Unk. σ^2

Now consider the case when we wish to detect a known signal with unknown parameters in AWGN with unknown variance. Suppose the observations $y \in \mathbb{R}^n$ are received from the classical linear model

$$y = Hx + w$$

with $x \in \mathbb{R}^{p \times 1}$ containing the unknown signal parameters and $W \sim \mathcal{N}(0, \sigma^2 I)$ with σ^2 unknown.

The hypotheses are given as

$$\begin{aligned} \mathcal{H}_0 &: x = 0 \text{ and } \sigma^2 > 0 \\ \mathcal{H}_1 &: x \neq 0 \text{ and } \sigma^2 > 0 \end{aligned}$$

GLRT for Classical Linear Model in AWGN with Unk. σ^2

Under the conditions of the previous slide, the GLRT decides $\mathcal{H}_1$ if

$$T(y) = \frac{n-p}{p} \frac{\hat{x}_1^\top H^\top H \hat{x}_1}{y^\top (I - H(H^\top H)^{-1} H^\top) y} > v$$

where $\hat{x}_1 = (H^\top H)^{-1} H^\top y$ is the MLE of x under $\mathcal{H}_1$. The decision statistic in this case is CFAR and

$$P_{\mathsf{fp}} = Q_U(v)$$
$$P_{\mathsf{D}} = Q_V(v)$$

where $Q_Z(x) = \int_x^\infty p_Z(t)\,dt$ is the tail probability of the random variable Z and

- $U \sim F_{p,n-p}$ denotes the F distribution with p numerator degrees of freedom and $n-p$ denominator degrees of freedom
- $V \sim F'_{p,n-p}(\lambda)$ denotes the non-central F distribution with p numerator degrees of freedom, $n-p$ denominator degrees of freedom, and non centrality parameter

$$\lambda = \frac{x^\top H^\top H x}{\sigma^2}$$

with x denoting the true value of the unknown signal parameters.

GLRT for Classical Linear Model in AWGN with Unk. σ^2

To provide some interpretation, let's rewrite the decision statistic

$$\begin{aligned}
T(y) &= \frac{n-p}{p}\frac{\hat{x}_1^\top H^\top H \hat{x}_1}{y^\top (I - H(H^\top H)^{-1}H^\top)y} \\
&= \frac{n-p}{p}\frac{((H^\top H)^{-1}H^\top y)^\top H^\top H (H^\top H)^{-1}H^\top y}{y^\top (I - H(H^\top H)^{-1}H^\top)y} \\
&= \frac{n-p}{p}\frac{y^\top H(H^\top H)^{-1}H^\top y}{y^\top (I - H(H^\top H)^{-1}H^\top)y} \\
&= \frac{n-p}{p}\frac{y^\top P y}{y^\top (I-P)y} = \frac{n-p}{p}\frac{\|Py\|^2}{\|(I-P)y\|^2}
\end{aligned}$$

where P is an orthogonal projection matrix onto the subspace of $\mathbb{R}^n$ spanned by the columns of H **(the signal subspace)** and $I - P$ is another orthogonal projection matrix onto the subspace of $\mathbb{R}^n$ orthogonal to the subspace spanned by the range of H **(the noise subspace)**.