ECE531 Screencast 13.5: GLRT with Unknown Signal Parameters and AWGN Noise Variance Example

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Problem Statement

Suppose we have the binary hypothesis testing problem

$$\begin{aligned}\mathcal{H}_0 &: Y = W \\ \mathcal{H}_1 &: Y = a1 + W\end{aligned}$$

with a unknown and $W \sim \mathcal{N}(0, \sigma^2 I)$ with σ^2 unknown.

The observations fit the classical linear model in this case

$$y = Hx + w$$

with $x = a$, $H = 1$, and $W \sim \mathcal{N}(0, \sigma^2 I)$ with σ^2 unknown.

The hypotheses can be restated as

$$\begin{aligned}\mathcal{H}_0 &: x = 0 \text{ and } \sigma^2 > 0 && \Leftrightarrow \mathcal{X}_0 = \{0\} \times (0, \infty) \\ \mathcal{H}_1 &: x \neq 0 \text{ and } \sigma^2 > 0 && \Leftrightarrow \mathcal{X}_1 = \{(-\infty, 0) \cup (0, \infty)\} \times (0, \infty)\end{aligned}$$

Solution (part 1 of 3)

We can apply our result that the GLRT decides $\mathcal{H}_1$ if

$$T(y) = \frac{n-p}{p} \frac{\hat{x}_1^\top H^\top H \hat{x}_1}{y^\top (I - H(H^\top H)^{-1} H^\top) y} > v$$

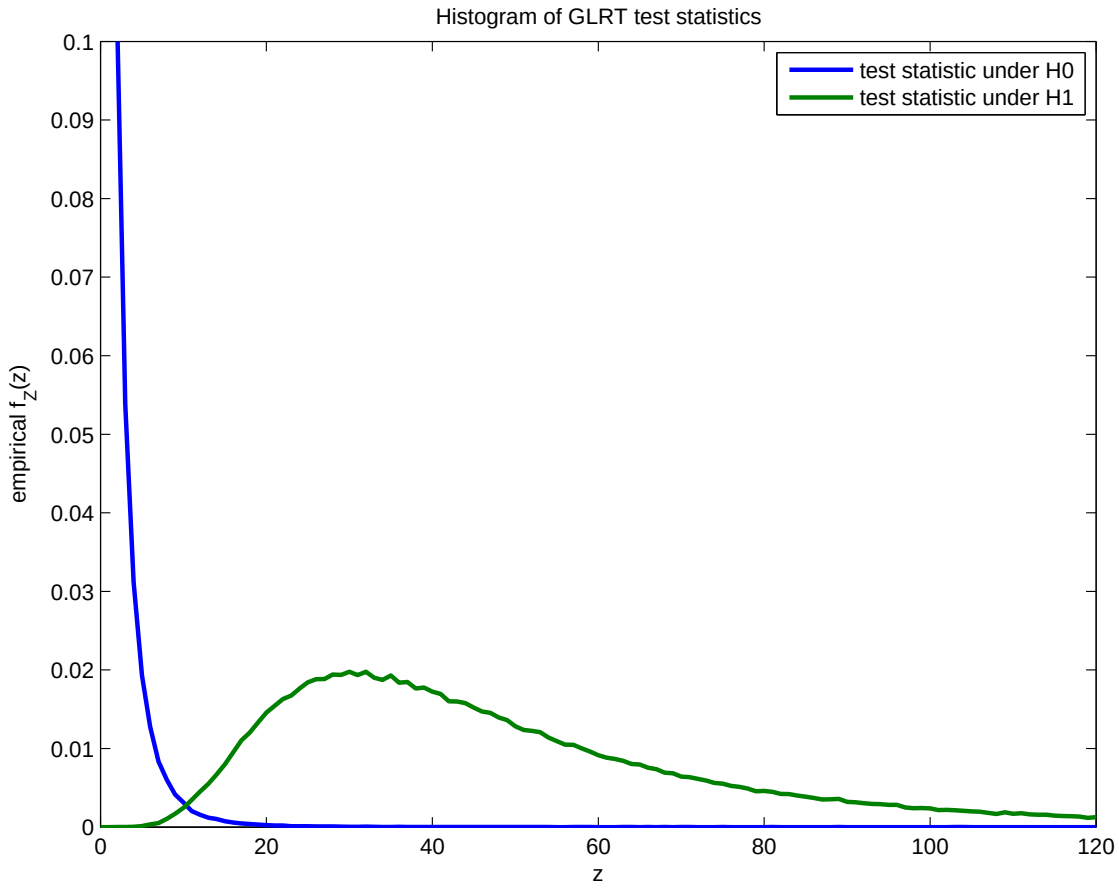
$$\Leftrightarrow (n-1) \frac{\frac{1}{n} y^\top H H^\top y}{y^\top (I - \frac{1}{n} H H^\top) y} > v$$

since $H^\top H = n$ and we have $p = 1$ unknown signal parameters. Note also that $HH^\top$ is a matrix of all ones, so we can further write

$$T(y) = \frac{\frac{n-1}{n} \left(\sum y_k\right)^2}{\|y\|^2 - \frac{1}{n} \left(\sum y_k\right)^2} > v$$

Test Statistics under $\mathcal{H}_0$ and $\mathcal{H}_1$

True parameter values: $\sigma^2 = 1$, $a = 2$.



Since we know this detector is CFAR, we know that the values of σ^2 and a do not affect the distribution of $Z = T(Y)$ under $\mathcal{H}_0$.

Solution (part 2 of 3)

To determine the decision threshold v, let's assume a false positive probability constraint of $\alpha = 0.01$. We can use the MATLAB function `finv` to compute the threshold to satisfy the false positive probability constraint:

```
v = finv(0.99,1,9)
```

where we used $0.99 = 1 - \alpha$ for the first argument since this function is based on the CDF, and the remaining arguments correspond to the numerator and denominator degrees of freedom, respectively. This yields $v = 10.5614$.

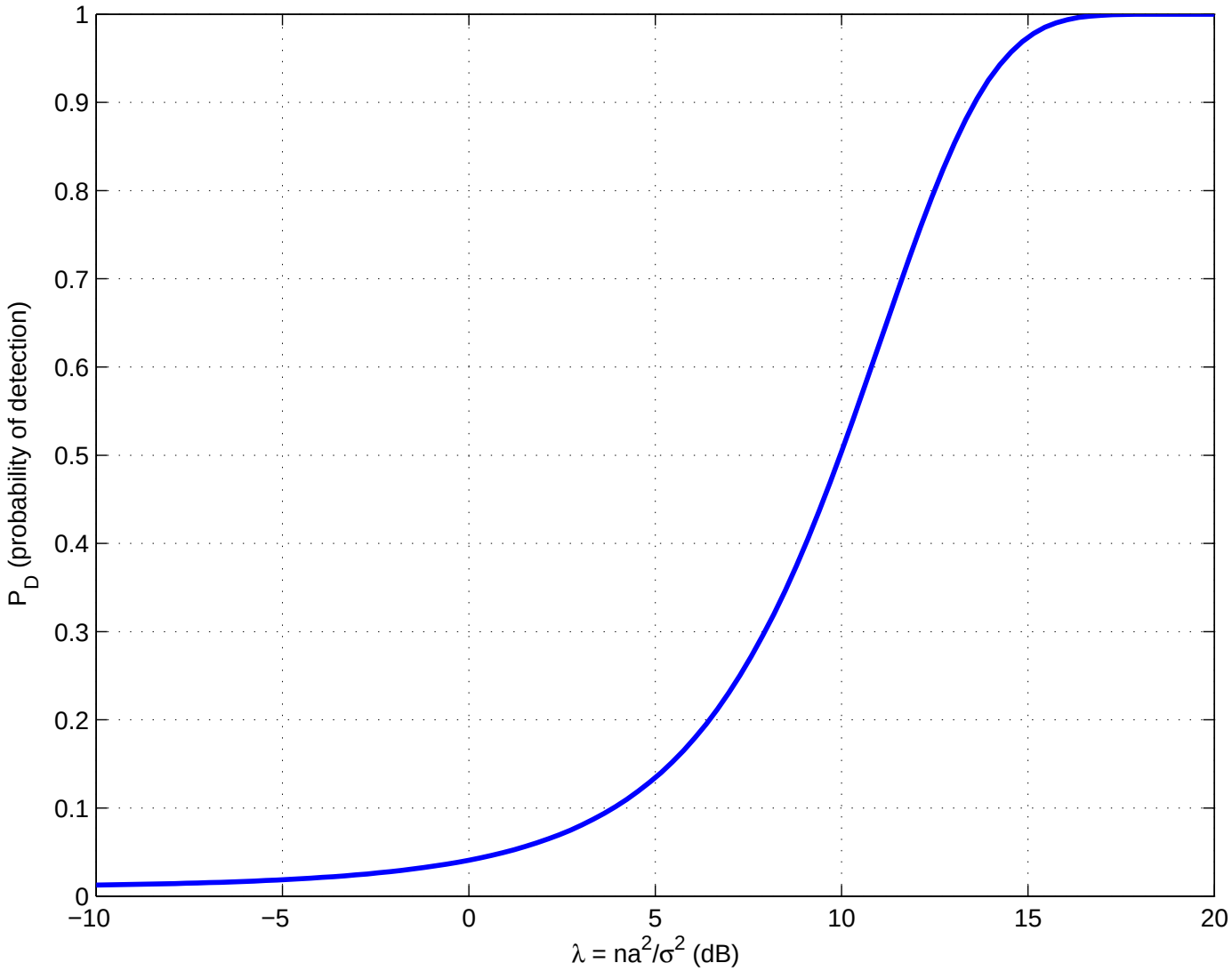
Solution (part 3 of 3)

To compute the probability of detection, we can use the MATLAB function `ncfcdf`. The probability of detection depends, of course, on the true parameter values for a and σ^2. The non centrality parameter in this case

$$\lambda = \frac{x^\top H^\top H x}{\sigma^2} = \frac{na^2}{\sigma^2}$$

which can be thought of as an integrated signal to noise ratio.

For a given λ we can compute the detection probability as

```
PD = 1-ncfcdf(v,1,9,lambda)
```

Probability of Detection vs. λ