ECE531 Screencast 2.1: Introduction to the Cramer-Rao Lower Bound (CRLB)

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Introduction

- Context: We are interested in understanding the performance of unbiased estimators under the squared error cost function.
- Squared error: Estimator variance $\mathrm{var}(\hat{\theta}(Y))$ determines performance.
- The CRLB gives a lower bound on the variance of an unbiased estimator: $\mathrm{var}(\hat{\theta}(Y)) \geq \mathrm{CRLB}$.

Why is this important?

- If a given unbiased estimator achieves the CRLB, i.e. $\mathrm{var}(\hat{\theta}(Y)) = \mathrm{CRLB}$, it must be the MVU estimator.
- A good lower bound also provides a benchmark by which we can compare the performance of different estimators.

Intuition: When Can We Expect Low Variance?

Recall our unknown parameter $\theta \in \Lambda$.

Suppose our parameter space $\Lambda = \mathbb{R}$ and we get a scalar observation distributed as $p_Y(y\,;\,\theta) = \mathcal{U}(0,1)$. What can we say about the performance of a good estimator $\hat{\theta}(y)$ in this case?

Suppose now that we get a scalar observation distributed as $p_Y(y\,;\,\theta) = \mathcal{U}(\theta - \epsilon, \theta + \epsilon)$ for some small value of ϵ. What can we say about the performance of a good estimator $\hat{\theta}(y)$ in this case?

Intuition: When Can We Expect Low Variance?

- The minimum achievable variance of an estimator is somehow related to the **sensitivity** of the density $p_Y(y\,;\,\theta)$ to changes in the parameter θ.
- If the density $p_Y(y\,;\,\theta)$ is **insensitive** to the parameter θ, then we can't expect even the MVU estimator to do very well.
- If the density $p_Y(y\,;\,\theta)$ is **sensitive** to changes in the parameter θ, then the achievable performance (minimum variance) should be better.
- Our notion of sensitivity:
  - Hold y fixed.
  - How "steep" is $p_Y(y\,;\,\theta)$ as we vary the parameter θ?
  - This steepness should somehow be averaged over the observations.
- Terminology: When we discuss $p_Y(y\,;\,\theta)$ with y fixed and θ as a variable, we call this a "likelihood function". It is not a valid pdf in θ. Recall that θ is not a random variable.

Example: Rayleigh Family $p_Y(y\,;\,\theta) = \frac{y}{\sigma^2} e^{\frac{-y^2}{\sigma^2}}$ with $\theta = \sigma$