ECE531 Screencast 2.4: Fisher Information for Vector Parameters

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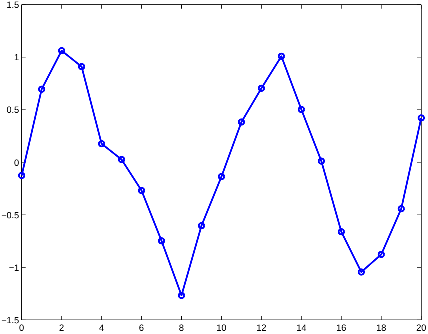
Vector Parameter Estimation Problems

In many problems, we have more than one parameter that we would like to estimate. For example,

$$Y_k = a\cos(\omega k + \phi) + W_k \text{ for } k = 0, 1, \ldots, n-1$$

where $a > 0$, $\phi \in (-\pi, \pi)$, and $\omega \in (0, \pi)$ are all non-random parameters and $W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$. In this problem $\theta = [a, \phi, \omega]$.



Fisher Information Matrix

Recall, in the scalar parameter case, the Fisher information was motivated by a computation of the mean squared relative slope of the likelihood function:

$$I(\theta) \quad := \quad \mathrm{E}\left[\left(\frac{\frac{\partial}{\partial\theta} p_Y(y\,;\,\theta)}{p_Y(y\,;\,\theta)}\right)^2\right] = \int_{\mathcal{Y}} \left(\frac{\partial}{\partial\theta} \ln p_Y(Y\,;\,\theta)\right)^2 p_Y(y\,;\,\theta)\,dy$$

In multiparameter problems, we are now concerned with the relative steepness of the likelihood function with respect to each of the parameters. A natural choice (assuming that all of the required derivatives exist) would be

$$I(\theta) = \mathrm{E}\left[(\nabla_\theta \ln p_Y(Y\,;\,\theta))\,(\nabla_\theta \ln p_Y(Y\,;\,\theta))^\top\right] \in \mathbb{R}^{m\times m}$$

where ∇_x is the gradient operator defined as

$$\nabla_x f(x) \quad := \quad \left[\frac{\partial}{\partial x_0} f(x), \ldots, \frac{\partial}{\partial x_{m-1}} f(x)\right]^\top.$$

Fisher Information Matrix

Let $p := p_Y(Y\,;\,\theta)$. The Fisher information matrix is then

$$I(\theta) = \begin{bmatrix} \mathrm{E}\left[\frac{\partial}{\partial\theta_0}\ln p \cdot \frac{\partial}{\partial\theta_0}\ln p\right] & \dots & \mathrm{E}\left[\frac{\partial}{\partial\theta_0}\ln p \cdot \frac{\partial}{\partial\theta_{m-1}}\ln p\right] \\ \mathrm{E}\left[\frac{\partial}{\partial\theta_1}\ln p \cdot \frac{\partial}{\partial\theta_0}\ln p\right] & \dots & \mathrm{E}\left[\frac{\partial}{\partial\theta_1}\ln p \cdot \frac{\partial}{\partial\theta_{m-1}}\ln p\right] \\ \vdots & \ddots & \vdots \\ \mathrm{E}\left[\frac{\partial}{\partial\theta_{m-1}}\ln p \cdot \frac{\partial}{\partial\theta_0}\ln p\right] & \dots & \mathrm{E}\left[\frac{\partial}{\partial\theta_{m-1}}\ln p \cdot \frac{\partial}{\partial\theta_{m-1}}\ln p\right] \end{bmatrix}$$

Note that the ijth element of the Fisher information matrix is given as

$$\boxed{I_{ij}(\theta) = \mathrm{E}\left[\frac{\partial}{\partial\theta_i}\ln p_Y\{Y\,;\,\theta\} \cdot \frac{\partial}{\partial\theta_j}\ln p_Y\{Y\,;\,\theta\}\right]}$$

hence we can say that $I(\theta)$ is symmetric.

Fisher Information Matrix

When the second derivatives all exist, we can write

$$\frac{\partial^2}{\partial\theta_i\theta_j}\ln p_Y(y\,;\,\theta) = \frac{\frac{\partial^2}{\partial\theta_i\theta_j}p_Y(y\,;\,\theta)}{p_Y(y\,;\,\theta)} - \frac{\frac{\partial}{\partial\theta_i}p_Y(y\,;\,\theta)}{p_Y(y\,;\,\theta)}\frac{\frac{\partial}{\partial\theta_j}p_Y(y\,;\,\theta)}{p_Y(y\,;\,\theta)}$$

and, under the theorem's assumptions, we can write

$$\mathrm{E}\left[\frac{\partial^2}{\partial\theta_i\theta_j}\ln p_Y(y\,;\,\theta)\right] = -\mathrm{E}\left[\frac{\partial}{\partial\theta_i}\ln p_Y(y\,;\,\theta)\cdot\frac{\partial}{\partial\theta_j}\ln p_Y(y\,;\,\theta)\right] = -I_{ij}(\theta).$$

Hence, we can say that

$$\boxed{I_{ij}(\theta) = -\mathrm{E}\left[\frac{\partial^2}{\partial\theta_i\theta_j}\ln p_Y(y\,;\,\theta)\right]}$$

This expression is often more convenient to compute than the former expression for $I_{ij}(\theta)$.

Fisher Information Matrix

Under the conditions of the theorem

$$\begin{aligned} \mathrm{E}\left[\frac{\partial}{\partial\theta_i}\ln p_Y\{Y\,;\,\theta\}\right] &= \int_{\mathcal{Y}} \frac{\frac{\partial}{\partial\theta_i}p_Y(y\,;\,\theta)}{p_Y(y\,;\,\theta)} p_Y(y\,;\,\theta)\,dy \\ &= \frac{\partial}{\partial\theta_i}\int_{\mathcal{Y}} p_Y(y\,;\,\theta)\,dy = 0 \end{aligned}$$

Hence

$$I_{ij}(\theta) = \mathrm{cov}\left\{\frac{\partial}{\partial\theta_i}\ln p_Y\{Y\,;\,\theta\}, \frac{\partial}{\partial\theta_j}\ln p_Y\{Y\,;\,\theta\}\right\}.$$

The Fisher information matrix $I(\theta)$ is a covariance matrix and is invertible if the unknown parameters are linearly independent.

Example: Fisher Information Matrix of Signal in AWGN

Many problems require the estimation of unknown signal parameters in additive white Gaussian noise. The observations in this case can be modeled as

$$Y_k \quad = \quad s_k(\theta) + W_k \text{ for } k = 0, 1, \dots, n-1$$

where $s_k(\theta) : \Lambda \mapsto \mathbb{R}$ is a deterministic signal with an unknown **vector** parameter θ and where $W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$. We assume σ^2 is a known parameter and that all of the regularity conditions are satisfied.

To compute the ijth element of the Fisher information matrix, we can write

$$\frac{\partial^2}{\partial\theta_i\theta_j} \ln p_Y(Y\,;\,\theta) = \frac{1}{\sigma^2} \sum_{k=0}^{n-1} \left\{ [Y_k - s_k(\theta)] \frac{\partial^2}{\partial\theta_i\theta_j} s_k(\theta) - \left(\frac{\partial}{\partial\theta_i} s_k(\theta)\right) \left(\frac{\partial}{\partial\theta_j} s_k(\theta)\right) \right\}$$

Since $\mathrm{E}[Y_k] = s_k(\theta)$, the ijth element of the FIM can be written as

$$I_{ij}(\theta) = -\mathrm{E}\left[\frac{\partial^2}{\partial\theta_i\theta_j} \ln p_Y(Y\,;\,\theta)\right] = \frac{1}{\sigma^2} \sum_{k=0}^{n-1} \left(\frac{\partial}{\partial\theta_i} s_k(\theta)\right) \left(\frac{\partial}{\partial\theta_j} s_k(\theta)\right)$$