ECE531 Screencast 2.5: Cramer-Rao Lower Bound for Vector Parameters

D. Richard Brown III

Worcester Polytechnic Institute

Cramer-Rao Lower Bound for Vector Parameters

The vector parameter Cramer-Rao lower bound (CRLB) can be simply expressed as

$$\boxed{\operatorname{cov}\left[\hat{\theta}(Y)\right] \geq I^{-1}(\theta)}$$

where $I^{-1}(\theta)$ is the inverse of the Fisher information matrix.

This is a matrix inequality here. When we write $A \geq B$, we mean the matrix $A - B$ is positive semidefinite.

Example: Estimating Amplitude and Phase

Consider the case where

$$Y_k = a\cos(\omega k + \phi) + W_k \text{ for } k = 0, 1, \dots, n-1$$

where ω is known, $a > 0$ and $\phi \in (-\pi, \pi)$ are unknown, and $W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$ with σ^2 known.

With $\theta = [a, \phi]^\top$, let's compute the Fisher information matrix:

$$\begin{aligned}
I_{00}(\theta) &= \frac{1}{\sigma^2}\sum_{k=0}^{n-1}\left(\frac{\partial}{\partial a}s_k(\theta)\right)^2 \\
&= \frac{1}{\sigma^2}\sum_{k=0}^{n-1}\cos^2(\omega k + \phi) = \frac{1}{\sigma^2}\sum_{k=0}^{n-1}\left(\frac{1}{2} + \cos(2(\omega k + \phi))\right) \approx \frac{n}{2\sigma^2} \\
I_{11}(\theta) &= \frac{1}{\sigma^2}\sum_{k=0}^{n-1}\left(\frac{\partial}{\partial \phi}s_k(\theta)\right)^2 = \frac{1}{\sigma^2}\sum_{k=0}^{n-1} a^2 \sin^2(\omega k + \phi) \approx \frac{na^2}{2\sigma^2} \\
I_{01}(\theta) &= \frac{1}{\sigma^2}\sum_{k=0}^{n-1}\left(\frac{\partial}{\partial a}s_k(\theta)\right)\left(\frac{\partial}{\partial \phi}s_k(\theta)\right) = -\frac{1}{\sigma^2}\sum_{k=0}^{n-1} a\sin(\omega k + \phi)\cos(\omega k + \phi) \approx 0
\end{aligned}$$

Example: Estimating Amplitude and Phase

Remarks:

- The Fisher information matrix in this example is (approximately)

$$I(\theta) = \frac{n}{2\sigma^2}\begin{bmatrix}1 & 0\\ 0 & a^2\end{bmatrix}$$

- Clearly $I(\theta)$ is positive definite when $a > 0$.
- Note that, since the observations are i.i.d., $I(\theta)$ satisfies the additive information property (as expected).
- We got lucky that the off-diagonal terms are (at least approximately) equal to zero here. The matrix inverse is easy to compute. This will not be true in general.

Example: Amplitude and Phase CRLB

We can compute the inverse of the Fisher information matrix easily:

$$I^{-1}(\theta) \;=\; \frac{2\sigma^2}{n}\begin{bmatrix}1 & 0\\ 0 & a^{-2}\end{bmatrix}$$

The CRLB in this is then simply

$$\mathrm{cov}\left[\hat{\theta}(Y)\right] \geq \frac{2\sigma^2}{n}\begin{bmatrix}1 & 0\\ 0 & a^{-2}\end{bmatrix}$$

The diagonal elements of $\mathrm{cov}\left[\hat{\theta}(Y)\right]$ reveal the minimum variance for each parameter: $\mathrm{var}_a\left[\hat{a}(Y)\right] \geq \frac{2\sigma^2}{n}$ and $\mathrm{var}_\phi\left[\hat{\phi}(Y)\right] \geq \frac{2\sigma^2}{na^2}$.