ECE531 Screencast 2.6: Cramer-Rao Lower Bound Example

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Problem Statement

Suppose Y is a scalar observation drawn from a parameterized Poisson distribution

$$p_Y(y\,;\,\theta) = \mathrm{Prob}(Y = y) = \frac{\theta^y e^{-\theta}}{y!}$$

for $y = 0, 1, 2, \dots$. We wish to estimate θ. Find the Fisher information $I(\theta)$ and the CRLB for estimating the scalar parameter θ. Can you find an MVU estimator that achieves the CRLB in this case?

Solution Step 1: Check the Conditions of the Theorem

To confirm the derivative exists for all $\theta \in \Lambda$ and all $y \in \mathcal{Y}$, we can compute

$$\frac{\partial}{\partial\theta} p_Y(y\,;\,\theta) = \frac{y\theta^{y-1}e^{-\theta} - \theta^y e^{-\theta}}{y!}.$$

You can take another partial derivative with respect to θ to confirm the second derivative exists as well.

We also need to check that $\mathrm{E}\left[\frac{\partial}{\partial\theta}\ln p_Y(Y\,;\,\theta)\right] = 0$. We can compute

$$\begin{aligned}
\mathrm{E}\left[\frac{\partial}{\partial\theta}\ln p_Y(Y\,;\,\theta)\right] = \int_{\mathcal{Y}} \frac{\partial}{\partial\theta} p_Y(y\,;\,\theta)\,dy &= \sum_{y=0}^{\infty} \frac{y\theta^{y-1}e^{-\theta} - \theta^y e^{-\theta}}{y!} \\
&= e^{-\theta}\sum_{y=0}^{\infty} \frac{y\theta^{y-1} - \theta^y}{y!} \\
&= e^{-\theta}\sum_{y=1}^{\infty} \frac{\theta^{y-1}}{(y-1)!} - e^{-\theta}\sum_{y=0}^{\infty}\frac{\theta^y}{y!} \\
&= e^{-\theta}e^{\theta} - e^{-\theta}e^{\theta} = 0
\end{aligned}$$

So this verifies the conditions of the theorem are satisfied.

Solution Step 2: Compute the Fisher Information

With the theorem conditions all confirmed, we can compute

$$\frac{\partial}{\partial\theta}\ln p_Y(y\,;\,\theta) = \frac{\partial}{\partial\theta}(-\theta + y\ln\theta) = -1 + \frac{y}{\theta},$$

and

$$\frac{\partial^2}{\partial\theta^2}\ln p_Y(y\,;\,\theta) = -\frac{y}{\theta^2} < 0.$$

The Fisher information is given by

$$I(\theta) = -\mathrm{E}\left\{\frac{\partial^2}{\partial\theta^2}\ln p_Y(Y\,;\,\theta)\right\} = \frac{\mathrm{E}\{Y\}}{\theta^2} = \frac{1}{\theta}.$$

Solution Step 3: Compute the CRLB and find MVU

From the Fisher information, CRLB is this case is simply

$$\mathrm{var}[\hat{\theta}(Y)] \geq \theta = \frac{1}{I(\theta)}.$$

To find an MVU estimator, let's try

$$\hat{\theta}(y) = y.$$

Since Y is Poisson, we have $\mathrm{E}\{\hat{\theta}(Y)\} = \theta$. So $\hat{\theta}(y)$ is an unbiased estimator of θ.

Since Y is Poisson, we also have $\mathrm{var}\{\hat{\theta}(Y)\} = \theta$. So $\hat{\theta}(y)$ achieves the CRLB and is MVU. You can also easily verify the CRLB attainability conditions here.