ECE531 Screencast 3.2: MVU Estimation Linear Model Example

D. Richard Brown III

Worcester Polytechnic Institute

Problem Statement

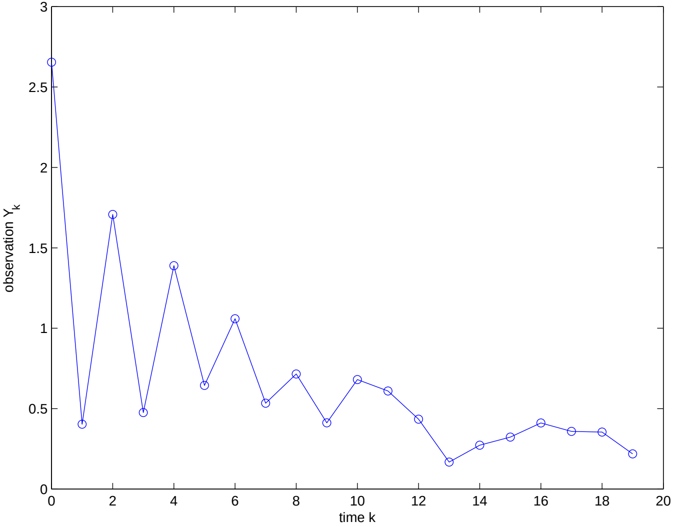
Kay 4.1 (rephrased with my notation): Given observations

$$Y_k = \sum_{i=1}^{p} \theta_i r_i^k + W_k$$

with r_i known for $i = 1, \dots, p$, and $W_k \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$, we wish to estimate the amplitudes $\theta = [\theta_1, \dots, \theta_p]^\top$.

Example Realization of the Observations

$r_1 = 0.9,\ r_2 = -0.8,\ \theta_1 = 1.5,\ \theta_2 = 1,\ \sigma^2 = 0.01,\ N = 20$



Solution Step 1: Put into Linear Model

Observations:

$$Y_k = \sum_{i=1}^{p} \theta_i r_i^k + W_k$$

We can put this into matrix/vector form as follows:

$$\underbrace{\begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{N-1} \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} 1 & 1 & \dots 1 \\ r_1 & r_2 & \dots r_p \\ \vdots & & \\ r_1^{N-1} & r_2^{N-1} & \dots r_p^{N-1} \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_p \end{bmatrix}}_{\theta} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{s} + \underbrace{\begin{bmatrix} W_0 \\ W_1 \\ \vdots \\ W_{N-1} \end{bmatrix}}_{W}$$

In this case, since the noise is i.i.d., we have $W \sim \mathcal{N}(0, C)$ with $C = \sigma^2 I$.

Solution Step 2: Compute MVU Estimator and CRLB

Once we have the linear model, we can just use our prior results

$$\hat{\theta}(y) = (H^\top H)^{-1} H^\top y$$

and

$$\mathrm{cov}(\hat{\theta}(Y)) = I^{-1}(\theta) = \sigma^2 (H^\top H)^{-1}$$

Given r_i and σ^2, one could compute these results by hand for $p = 1$ and $p = 2$. For $p > 2$, you will probably need MATLAB.

Solution Step 3: Specific Case

Let's work this out for the specific case $p = 2$, $r_1 = 1$, $r_2 = -1$, and N an even number. In this case we have

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } H^\top H = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} = NI$$

Hence

$$\hat{\theta}(y) = (H^\top H)^{-1} H^\top y = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -1 & 1 & -1 & \dots & 1 & -1 \end{bmatrix} y$$

and

$$\text{cov}(\hat{\theta}(Y)) = \sigma^2 (H^\top H)^{-1} = \frac{\sigma^2}{N} I.$$