ECE531 Screencast 3.5: The RBLS Theorem

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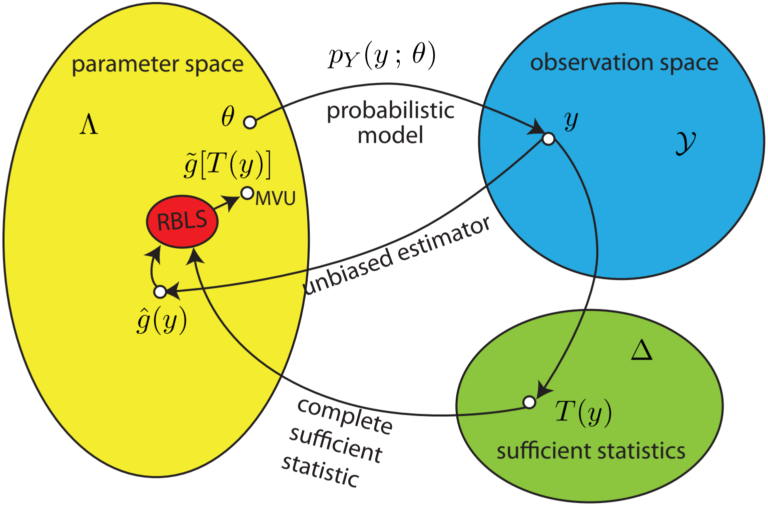
Conceptual Model



Rao-Blackwell-Lehmann-Sheffe Theorem

Theorem

If $\hat{g}(y)$ is any unbiased estimator of θ and T is a sufficient statistic for the family $\{p_Y(y\,;\,\theta)\,;\,\theta \in \Lambda\}$, then

$$\tilde{g}[T(y)] \quad := \quad \mathrm{E}\left[\hat{g}(Y)\,|\,T(Y) = T(y)\right]$$

is

- *A valid estimator of θ (not a function of θ)*
- *An unbiased estimator of θ.*
- *Of lesser or equal variance than that of $\hat{g}(y)$ for all $\theta \in \Lambda$*

Additionally, if T is complete, then $\tilde{g}[T(y)]$ is an MVU estimator of of θ.

Example: Estimating a Constant in White Gaussian Noise

Suppose $\theta \in \mathbb{R}$ and $Y_k = \theta + W_k$ with $W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$ for $k = 0, \dots, n-1$. Then the joint distribution

$$p_Y(y\,;\,\theta) \quad = \quad \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{ \frac{-1}{2\sigma^2} \sum_{k=0}^{n-1} (y_k - \theta)^2 \right\}.$$

Let $T(y) = \frac{1}{n} \sum_{k=0}^{n-1} y_k$. We know this is a complete sufficient statistic. Let's apply the RBLS theorem to find the MVU estimator...

- We could choose the unbiased estimator $\hat{g}(y) = y_0$.
- Now we need to compute

$$\tilde{g}[T(y)] := \mathrm{E}\left[\hat{g}(Y) \,|\, T(Y) = T(y)\right] = \mathrm{E}\left[Y_0 \,|\, \frac{1}{n} \sum Y_k = \frac{1}{n} \sum y_k \right]$$

- To solve this, we can use a standard formula for the conditional expectation of a jointly Gaussian random variable...

Example (continued)

- Suppose $Z = [X, Y]^\top$ is jointly Gaussian distributed. We know that

$$\mathrm{E}[X|Y=y] \;=\; \mathrm{E}[X] + \frac{\mathsf{cov}(X,Y)}{\mathsf{var}(Y)}(y - \mathrm{E}[Y]).$$

- In our problem, letting $\bar{Y} = \frac{1}{n}\sum_{k=0}^{n-1} Y_k$, we can use this result to write

$$\begin{aligned}\mathrm{E}\left[Y_0 \,|\, \bar{Y} = t\right] &= \mathrm{E}[Y_0] + \frac{\mathsf{cov}(Y_0, \bar{Y})}{\mathsf{var}(\bar{Y})}\left(t - \mathrm{E}[\bar{Y}]\right) \\ &= \theta + \frac{\sigma^2}{\sigma^2}\left(\frac{1}{n}\sum y_k - \theta\right) \\ &= \frac{1}{n}\sum y_k\end{aligned}$$

- Hence $\hat{\theta}_{\mathsf{mvu}}(y) = \frac{1}{n}\sum y_k$ is an MVU estimator of θ (as expected).