ECE531 Screencast 4.6: MLE Example: Kay vI:7.8

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Kay vI Problem 7.8

We observe N independent coin tosses with $Y_k = 1$ with probability θ and $Y_k = 0$ with probability $1 - \theta$. Find the MLE of the parameter θ.

Note this is a vector observation here. Unlike some problems you may have seen earlier, the observation $Y \in \{0, 1\}^N$ is a vector of ones and zeros drawn from the i.i.d. coin flips.

Solution Step 1

A good first step is to write the joint pdf (really pmf) of the i.i.d. observations as

$$
\begin{aligned}
p_Y(y\,;\,\theta) &= \prod_{k=0}^{N-1} \theta^{y_k}(1-\theta)^{(1-y_k)} \\
&= \theta^{\sum_k y_k}(1-\theta)^{\sum_k (1-y_k)} \\
&= \theta^{N\bar{y}}(1-\theta)^{N-N\bar{y}}
\end{aligned}
$$

with $\bar{y} = \frac{1}{N}\sum_k y_k$.

Solution Step 2

To find the MLE of θ, we compute

$$\begin{aligned}
\frac{\partial}{\partial\theta} \ln p_Y(y\,;\,\theta) &= \frac{\partial}{\partial\theta} \left\{ N\bar{y} \ln\theta + (N - N\bar{y}) \ln(1-\theta) \right\} \\
&= \frac{N\bar{y}}{\theta} + \frac{N - N\bar{y}}{1-\theta} \cdot (-1) \\
&= \frac{N\bar{y}}{\theta} - \frac{N - N\bar{y}}{1-\theta}
\end{aligned}$$

We need to set this equal to zero and solve for θ. If we do this, we get

$$\begin{aligned}
& N\bar{y}(1-\theta) - (N - N\bar{y})\theta = 0 \\
\Leftrightarrow \quad & N\bar{y} - N\bar{y}\theta - N\theta + N\bar{y}\theta = 0 \\
\Leftrightarrow \quad & \theta = \bar{y}.
\end{aligned}$$

In other words, the unique solution of the likelihood equation is just the sample mean of the observation vector y.

Solution Step 3

To confirm this solution is the MLE, we need to confirm it achieves the maximum of the likelihood function. We can compute the second derivative

$$\frac{\partial^2}{\partial\theta^2}\ln p_Y(y\,;\,\theta) = \frac{\partial}{\partial\theta}\left\{\frac{N\bar{y}}{\theta} - \frac{N-N\bar{y}}{1-\theta}\right\}$$
$$= \frac{-N\bar{y}}{\theta^2} - \frac{N-N\bar{y}}{(1-\theta)^2}.$$

Substitute in $\theta = \bar{y}$ to get

$$\frac{\partial^2}{\partial\theta^2}\ln p_Y(y\,;\,\theta) = \frac{-N\bar{y}}{\bar{y}^2} - \frac{N-N\bar{y}}{(1-\bar{y})^2}$$
$$= -\left(\frac{N}{\bar{y}} + \frac{N}{1-\bar{y}}\right)$$

which is strictly less than zero for all $0 < \theta < 1$. Hence $\ln p_Y(y\,;\,\theta)$ is a concave function and $\hat{\theta}_{\mathsf{ML}}(y) = \bar{y}$ must be the MLE.