ECE531 Screencast 5.6: An Example of Bayesian Estimation for the Linear Gaussian Model

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Estimation of Signal Amplitude in Gaussian Noise

Suppose now that we observe

$$Y_k = \Theta s_k + W_k \qquad k = 0, \dots, n-1$$

where $s = [s_0, \dots, s_{n-1}]$ is known, $W \sim \mathcal{N}(0, \Sigma_W)$, and $\Theta \sim \mathcal{N}(\mu, v^2)$. Note that Θ is a scalar parameter. As usual, we assume the parameter and noise are independent.

Let's derive the MMSE/MMAE/MAP Bayesian estimators for the unknown parameter Θ...

Solution Step 1: Write Linear Gaussian Model

We want to write the observations in this form:

$$Y = H\Theta + W$$

Since

$$Y_k = \Theta s_k + W_k \text{ for } k = 0, \dots, N-1$$

we can put this into matrix/vector form as follows:

$$\underbrace{\begin{bmatrix} Y_0 \\ \vdots \\ Y_{n-1} \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} s_0 \\ \vdots \\ s_{n-1} \end{bmatrix}}_{H} \Theta + \underbrace{\begin{bmatrix} W_0 \\ \vdots \\ W_{N-1} \end{bmatrix}}_{W}$$

Solution Step 2: Compute Conditional Expectation (1 of 2)

We can use our prior result:

$$\begin{aligned}
\mathrm{E}[\Theta \,|\, Y = y] &= \mu_\Theta + \Sigma_\Theta H^\top \left(H \Sigma_\Theta H^\top + \Sigma_W \right)^{-1} (y - H\mu_\Theta) \\
&= \mu + v^2 H^\top \left(v^2 H H^\top + \Sigma_W \right)^{-1} (y - H\mu)
\end{aligned}$$

where we've substituted $\Sigma_\Theta = v^2$ and $\mu_\Theta = \mu$. The difficult part here is the matrix inversion. Let's use the matrix inversion lemma again

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}$$

With $A = \Sigma_W$, $B = H$, $C = H^\top$, and $D = -1/v^2$, we have

$$\begin{aligned}
(A - BD^{-1}C)^{-1} &= \left(H \Sigma_\Theta H^\top + \Sigma_W \right)^{-1} \\
&= \Sigma_W^{-1} + \Sigma_W^{-1} H \left(\frac{-1}{v^2} - H^\top \Sigma_W^{-1} H \right)^{-1} H^\top \Sigma_W^{-1}
\end{aligned}$$

Note the quantity to be inverted now is just a scalar.

Solution Step 2: Compute Conditional Expectation (2 of 2)

Using the result of the matrix inversion lemma, we can write

$$\begin{aligned}
\mathrm{E}[\Theta \,|\, Y=y] &= \mu + v^2 H^\top \left(\Sigma_W^{-1} - \Sigma_W^{-1} H \left(\frac{1}{v^2} + H^\top \Sigma_W^{-1} H \right)^{-1} H^\top \Sigma_W^{-1} \right) (y - H\mu) \\
&= \mu + v^2 H^\top \left(\Sigma_W^{-1} - \frac{\Sigma_W^{-1} H H^\top \Sigma_W^{-1}}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H} \right) (y - H\mu) \\
&= \mu + v^2 \left(H^\top \Sigma_W^{-1} - \frac{H^\top \Sigma_W^{-1} H H^\top \Sigma_W^{-1}}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H} \right) (y - H\mu) \\
&= \mu + v^2 \left(1 - \frac{H^\top \Sigma_W^{-1} H}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H} \right) H^\top \Sigma_W^{-1} (y - H\mu) \\
&= \mu + v^2 \left(\frac{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H - H^\top \Sigma_W^{-1} H}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H} \right) H^\top \Sigma_W^{-1} (y - H\mu) \\
&= \mu + \frac{H^\top \Sigma_W^{-1}}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H} (y - H\mu)
\end{aligned}$$

This is the MMSE/MMAE/MAP estimator.

Solution Step 3: Compute Conditional Covariance

To quantify the performance of our estimator, we can compute

$$
\begin{aligned}
\text{cov}[\Theta \,|\, Y = y] &= \Sigma_\Theta - \Sigma_\Theta H^\top \left(H \Sigma_\Theta H^\top + \Sigma_W \right)^{-1} H \Sigma_\Theta \\
&= v^2 - v^2 H^\top \left(v^2 H H^\top + \Sigma_W \right)^{-1} H v^2 \\
&= v^2 - v^2 H^\top \left(\Sigma_W^{-1} - \frac{\Sigma_W^{-1} H H^\top \Sigma_W^{-1}}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H} \right) H v^2 \\
&= v^2 - v^4 \left(H^\top \Sigma_W^{-1} H - \frac{H^\top \Sigma_W^{-1} H H^\top \Sigma_W^{-1} H}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H} \right) \\
&= v^2 - v^4 \left(\frac{\frac{1}{v^2} H^\top \Sigma_W^{-1} H}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H} \right) \\
&= v^2 \left(1 - \frac{H^\top \Sigma_W^{-1} H}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H} \right) \\
&= \frac{1}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H}
\end{aligned}
$$

Summary and Remarks

$$\hat{\theta}_{\text{mmse}}(y) = \mu + \frac{H^\top \Sigma_W^{-1}}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H}(y - H\mu)$$

$$\text{MMSE} = \frac{1}{\frac{1}{v^2} + H^\top \Sigma_W^{-1} H}$$

If the noise is i.i.d. then $\Sigma_W = \sigma^2 I$ and we have

$$\hat{\theta}_{\text{mmse}}(y) = \mu + \frac{\frac{1}{\sigma^2} H^\top}{\frac{1}{v^2} + \frac{1}{\sigma^2} H^\top H}(y - H\mu)$$

$$\text{MMSE} = \frac{1}{\frac{1}{v^2} + \frac{1}{\sigma^2} H^\top H}$$

Remarks:

- $v^2 \to 0$, $v^2 \to \infty$?
- $\sigma^2 \to 0$, $\sigma^2 \to \infty$?