ECE531 Screencast 6.6: Sequential LMMSE Example

D. Richard Brown III

Worcester Polytechnic Institute

Problem Statement

Kay v1 12.13: This problem is based on Example 12.1 where we have a constant $\Theta \in \mathbb{R}$ observed in WGN with

$$Y_k = \Theta + W_k$$

with $Y_k \sim \mathcal{U}[-A_0, A_0]$ and $W_k \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$ and independent of Θ. The problem is to find the sequential LMMSE estimator and confirm that it agrees with the batch LMMSE estimator given (12.9) as

$$\hat{\theta}_{\mathsf{LMMSE}}(y_0, \dots, y_{N-1}) = \frac{\frac{A_0^2}{3}}{\frac{A_0^2}{3} + \frac{\sigma^2}{N}} \bar{y}$$

where $\bar{y}$ is the usual sample mean.

Note that this is problem fits in our linear model with $h[k] \equiv 1$.

Solution Step 1: Initial Estimates

First, lets compute the estimate and the error variance before we receive any observations.

Before any observations arrive, the LMMSE estimate of Θ is just its mean. Hence

$$\hat{\theta}_{\mathsf{LMMSE}}[-1] = 0$$

and the error covariance of this estimate is just the variance of Θ, i.e.

$$\Sigma[-1] = \mathrm{var}(\Theta) = \frac{4A_0^2}{12} = \frac{A_0^2}{3}$$

Solution Step 2: Sequential LMMSE Recursion

Now when observations y_k arrive at times $k = 0, 1, \dots,$ we have the general recursion (recall $h = 1$):

$$K[k] = \frac{\Sigma[k-1]}{\sigma^2 + \Sigma[k-1]}$$

and

$$\begin{aligned} \hat{\theta}_{\text{LMMSE}}[k] &= \hat{\theta}_{\text{LMMSE}}[k-1] + K[k]\left(y_k - \hat{\theta}_{\text{LMMSE}}[k-1]\right) \\ &= \hat{\theta}_{\text{LMMSE}}[k-1] + \frac{\Sigma[k-1]}{\sigma^2 + \Sigma[k-1]}\left(y_k - \hat{\theta}_{\text{LMMSE}}[k-1]\right) \end{aligned}$$

with estimation error variance

$$\Sigma[k] = (1 - K[k])\Sigma[k-1]$$

Solution Step 3: Evolution of K and Σ

Let's look at how K and Σ evolve...

$$K[0] = \frac{\Sigma[-1]}{\sigma^2 + \Sigma[-1]} = \frac{\frac{A_0^2}{3}}{\sigma^2 + \frac{A_0^2}{3}}$$

$$\Sigma[0] = (1 - K[0])\Sigma[-1] = \frac{\sigma^2}{\sigma^2 + \frac{A_0^2}{3}} \cdot \frac{A_0^2}{3}$$

$$K[1] = \frac{\Sigma[0]}{\sigma^2 + \Sigma[0]} = \frac{\frac{\sigma^2 \frac{A_0^2}{3}}{\sigma^2 + \frac{A_0^2}{3}}}{\sigma^2 + \frac{\sigma^2 \frac{A_0^2}{3}}{\sigma^2 + \frac{A_0^2}{3}}} = \frac{\frac{A_0^2}{3}}{\sigma^2 + 2\frac{A_0^2}{3}}$$

$$\Sigma[1] = (1 - K[1])\Sigma[0] = \left(1 - \frac{\frac{A_0^2}{3}}{\sigma^2 + 2\frac{A_0^2}{3}}\right) \frac{\sigma^2 \frac{A_0^2}{3}}{\sigma^2 + \frac{A_0^2}{3}} = \frac{\sigma^2 \frac{A_0^2}{3}}{\sigma^2 + 2\frac{A_0^2}{3}}$$

Solution Step 4: Direct expression for $\hat{\theta}_{\text{LMMSE}}[k]$

Based on these results, one can prove by induction that

$$K[k] = \frac{\frac{A_0^2}{3}}{\sigma^2 + (k+1)\frac{A_0^2}{3}}$$

$$\Sigma[k] = \frac{\sigma^2 \frac{A_0^2}{3}}{\sigma^2 + (k+1)\frac{A_0^2}{3}}$$

Now lets plug these results into the estimator equation:

$$\begin{aligned}
\hat{\theta}_{\text{LMMSE}}[k] &= \hat{\theta}_{\text{LMMSE}}[k-1] + K[k]\left(y_k - \hat{\theta}_{\text{LMMSE}}[k-1]\right) \\
&= \hat{\theta}_{\text{LMMSE}}[k-1] + \frac{\frac{A_0^2}{3}}{\sigma^2 + (k+1)\frac{A_0^2}{3}}\left(y_k - \hat{\theta}_{\text{LMMSE}}[k-1]\right) \\
&= \left(\frac{\sigma^2 + k\frac{A_0^2}{3}}{\sigma^2 + (k+1)\frac{A_0^2}{3}}\right)\hat{\theta}_{\text{LMMSE}}[k-1] + \frac{\frac{A_0^2}{3}}{\sigma^2 + (k+1)\frac{A_0^2}{3}} y_k
\end{aligned}$$

Solution Step 5: Evolution of $\hat{\theta}_{\text{LMMSE}}[k]$

Given $\hat{\theta}_{\text{LMMSE}}[-1] = 0$ and

$$\hat{\theta}_{\text{LMMSE}}[k] = \left(\frac{\sigma^2 + k\frac{A_0^2}{3}}{\sigma^2 + (k+1)\frac{A_0^2}{3}} \right) \hat{\theta}_{\text{LMMSE}}[k-1] + \frac{\frac{A_0^2}{3}}{\sigma^2 + (k+1)\frac{A_0^2}{3}} y_k$$

we can write

$$\hat{\theta}_{\text{LMMSE}}[0] = \frac{\frac{A_0^2}{3}}{\sigma^2 + \frac{A_0^2}{3}} y_0$$

$$\hat{\theta}_{\text{LMMSE}}[1] = \left(\frac{\sigma^2 + \frac{A_0^2}{3}}{\sigma^2 + 2\frac{A_0^2}{3}} \right) \frac{\frac{A_0^2}{3}}{\sigma^2 + \frac{A_0^2}{3}} y_0 + \frac{\frac{A_0^2}{3}}{\sigma^2 + 2\frac{A_0^2}{3}} y_1 = \frac{\frac{A_0^2}{3}}{\frac{\sigma^2}{2} + \frac{A_0^2}{3}} \left(\frac{1}{2}(y_0 + y_1) \right)$$

Based on these results, one can then prove by induction that

$$\hat{\theta}_{\text{LMMSE}}[k] = \frac{\frac{A_0^2}{3}}{\frac{\sigma^2}{k+1} + \frac{A_0^2}{3}} \left(\frac{1}{k+1} \sum_{j=0}^{k} y_j \right)$$

which is equivalent to the batch solution in (12.9) when $k = n - 1$.