ECE531 Screencast 7.2: Discrete Time Model for Dynamic Parameters

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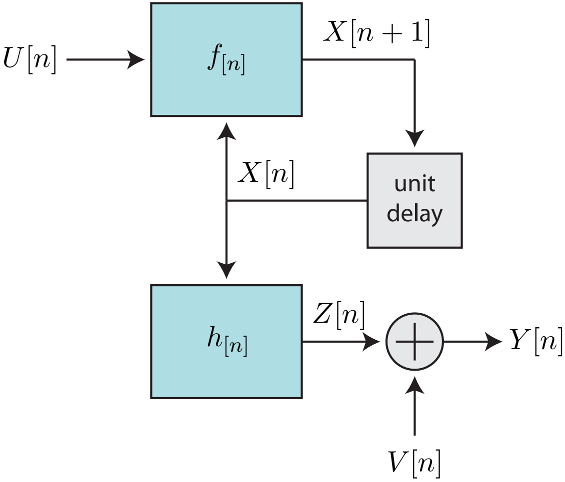
Discrete Time Model for Dynamic Parameters



The time index is denoted as $n = 0, 1, \ldots$. All vectors are considered to be random unless otherwise specified.

$$
\begin{aligned}
U[n] &\in \mathbb{R}^s \\
X[n] &\in \mathbb{R}^m \\
Z[n] &\in \mathbb{R}^k \\
V[n] &\in \mathbb{R}^k \\
Y[n] &\in \mathbb{R}^k \\
f_{[n]} &: \mathbb{R}^s \times \mathbb{R}^m \mapsto \mathbb{R}^m \\
h_{[n]} &: \mathbb{R}^m \mapsto \mathbb{R}^k
\end{aligned}
$$

Notation and Terminology

- $U[n]$ is the dynamical system "input", $n = 0, 1, \ldots$. This input is usually random and is also called the "process noise".
- $X[n]$ is the "state" of the dynamical system, $n = 0, 1, \ldots$. This is what we want to estimate.
- $Z[n]$ is the dynamical system "output", $n = 0, 1, \ldots$.
- $f_{[n]}$ is a time varying function that updates the state based on the current state and the current input, i.e.

$$X[n+1] = f_{[n]}(X[n], U[n])$$

- $h_{[n]}$ is a time varying function that generates the current output based on the current state, i.e.

$$Z[n] = h_{[n]}(X[n])$$

- $V[n]$ is the "measurement noise" $n = 0, 1, \ldots$.
- $Y[n]$ is the "observation" $n = 0, 1, \ldots$.

Example: One-Dimensional Motion

Suppose we have a particle moving on a line with position x and velocity v updated according to

$$\begin{aligned} x[n+1] &= x[n] + Tv[n] \\ v[n+1] &= v[n] + Ta[n] \end{aligned}$$

where $a[n]$ represents the piecewise constant acceleration and T is the time between samples. The implicit assumption here is that T is small enough such that the piecewise constant approximation holds.

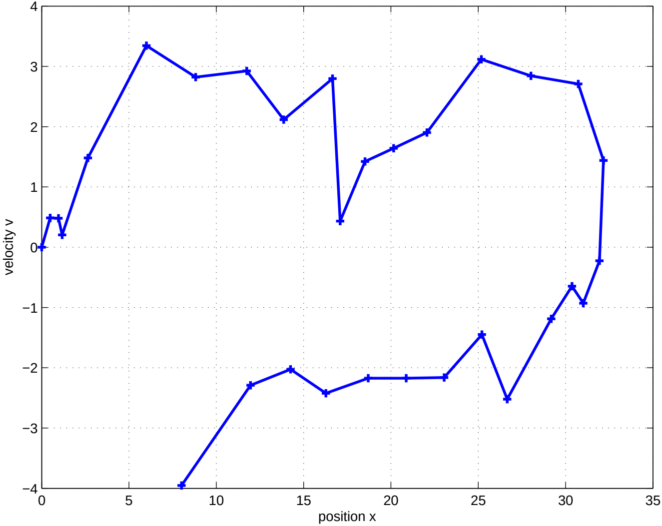
- The state $X = [x, v]^\top$.
- The input $U = a$.
- The state update equation can be written as

$$X[n+1] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} X[n] + \begin{bmatrix} 0 \\ T \end{bmatrix} U[n]$$

- Suppose we observe the position of the particle in noise. Then

$$Y[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} X[n] + V[n]$$

Example: One-Dimensional Motion (White Gaussian Input)



Linear Dynamical Model

For now, we are going to restrict our attention to systems with state update equations and output equations of the form

$$\begin{aligned} X[n+1] &= F[n]X[n] + G[n]U[n] \qquad n = 0, 1, \dots \\ Y[n] &= H[n]X[n] + V[n] \qquad n = 0, 1, \dots \end{aligned}$$

where, for each n, $F[n] \in \mathbb{R}^{m \times m}$, $G[n] \in \mathbb{R}^{m \times s}$, and $H[n] \in \mathbb{R}^{k \times m}$.

- We've already seen that one-dimensional motion fits within this linear model.
- The same is true for two- and three-dimensional motion and lots of other real-world dynamic systems.
- Many nonlinear systems can approximately fit in this model by linearizing f and h around a nominal state (Taylor series expansion).

Gaussian Input and Measurement Noise

We will only consider systems in which the input (process noise) sequence $U[0], U[1], \dots$ and the measurement noise sequence $V[0], V[1], \dots$ are independent sequences of independent zero mean Gaussian random vectors, i.e.

$$
\begin{aligned}
\mathrm{E}\{U[k]\} &= 0 \\
\mathrm{E}\{U[k]U^\top[j]\} &= \begin{cases} Q[k] & k = j \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{E}\{V[k]\} &= 0 \\
\mathrm{E}\{V[k]V^\top[j]\} &= \begin{cases} R[k] & k = j \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{E}\{U[k]V^\top[j]\} &= 0 \qquad \text{for all } j \text{ and } k
\end{aligned}
$$

We also assume that the initial state $X[0] \sim \mathcal{N}(m[0], \Sigma[0])$ is a Gaussian random vector independent of $U[0], U[1], \dots$ and $V[0], V[1], \dots$.