ECE531 Screencast 7.4: The Extended Kalman Filter

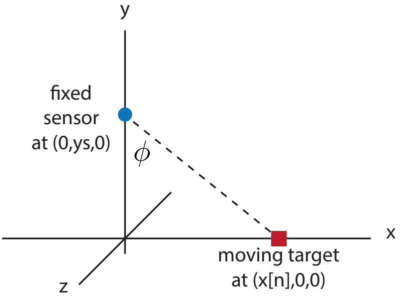
D. Richard Brown III

Worcester Polytechnic Institute

Nonlinear Dynamical Model Example

Many tracking problems are based on estimating the coordinates of a target given measurements of range and/or bearing.

A simple example: Suppose we have a target moving on a line with state $X = [x, v]^\top$ where $(x, 0, 0)$ is the position of the target and v is the velocity of the particle on the x-axis with acceleration process noise $U[n]$



State update:

$$X[n+1] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} X[n] + \begin{bmatrix} 0 \\ T \end{bmatrix} U[n]$$

Suppose we have a sensor at position $(0, y_s, 0)$ that can measure the **angle** of the target. Measurement:

$$Y[n] = \tan^{-1}\left(\frac{x[n]}{y_s}\right) + V[n]$$

Extended Kalman Filter Dynamical Model Assumptions

We now consider systems of the form

$$X[n+1] = f_{[n]}(X[n]) + G[n]U[n] \qquad n = 0, 1, \dots$$
$$Y[n] = h_{[n]}(X[n]) + V[n] \qquad n = 0, 1, \dots$$

where $f_{[n]}$ and $h_{[n]}$ can be nonlinear functions of $X[n]$. These nonlinear functions can be time-varying but must be differentiable with respect to the state.

The assumptions about Gaussian input and measurement noise from the Kalman filter still apply here. The only change is that the state update and/or measurement equations can be nonlinear in the state.

Extended Kalman Filter: Main Idea

The main idea of the EKF is to linearize the nonlinear state update and/or measurement functions around our current estimates/predictions of the state. To do this, we use a vector Taylor series expansion.

First-order vector Taylor series expansion around $x = x_0$:

$$f(x) \approx f(x_0) + \underbrace{J_x(f(x))}_{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix}} \Big|_{x=x_0}(x - x_0)$$

The associated Jacobians for the EKF are denoted as:

$$\bar{F}[n] = J_x(f_{[n]}(x))\big|_{x=\hat{X}[n|n]}$$

$$\bar{H}[n+1] = J_x(h_{[n+1]}(x))\big|_{x=\hat{X}[n+1|n]}$$

Approximate State-Update and Observation Equations

Using a first-order Taylor series approximation, we have

$$X[n+1] \approx f_{[n]}(\hat{X}[n|n]) + \bar{F}[n]\left(X[n] - \hat{X}[n|n]\right) + G[n]U[n]$$

$$Y[n+1] \approx h_{[n+1]}(\hat{X}[n+1|n]) + \bar{H}[n+1]\left(X[n+1] - \hat{X}[n+1|n]\right) + V[n+1]$$

which can be rewritten as

$$X[n+1] \approx \bar{F}[n]X[n] + G[n]U[n] + \left[f_{[n]}(\hat{X}[n|n]) - \bar{F}[n]\hat{X}[n|n]\right]$$

$$Y[n+1] \approx \bar{H}[n+1]X[n+1] + V[n+1] + \left[h_{[n+1]}(\hat{X}[n+1|n]) - \bar{H}[n+1]\hat{X}[n+1|n]\right]$$

Note this is like our linear model from before with the difference being that there are some known terms added to the state update and observation.

In the earlier example, we have $\bar{F}[n] = F[n] = f_{[n]}$ since the state-update is linear and $h_{[n]} = \tan^{-1}(x/y_s)$. We can compute this Jacobian

$$\bar{H}[n] = \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial v} \end{bmatrix}_{[x,v]=[\hat{x}[n+1|n],\hat{v}[n+1|n]} = \begin{bmatrix} \frac{1}{y_s\left(\frac{(\hat{x}[n+1|n])^2}{y_s^2}+1\right)} & 0 \end{bmatrix}$$

EKF: Summary of General Recursion

Initialization (predictions):

$$
\begin{aligned}
\hat{X}[0 \,|\, -1] &= m[0] \\
\Sigma[0 \,|\, -1] &= \Sigma[0]
\end{aligned}
$$

Recursion, beginning with $\ell = 0$:

$$
\begin{aligned}
K[\ell] &= \Sigma[\ell \,|\, \ell-1]\bar{H}^\top[\ell]\left(\bar{H}[\ell]\Sigma[\ell \,|\, \ell-1]\bar{H}^\top[\ell] + R[\ell]\right)^{-1} \\
\hat{X}[\ell \,|\, \ell] &= \hat{X}[\ell \,|\, \ell-1] + K[\ell]\left(Y[\ell] - h_{[\ell]}\hat{X}[\ell \,|\, \ell-1]\right) \\
\Sigma[\ell \,|\, \ell] &= \Sigma[\ell \,|\, \ell-1] - K[\ell]\bar{H}[\ell]\Sigma[\ell \,|\, \ell-1] \\
\hat{X}[\ell+1 \,|\, \ell] &= f_{[\ell]}\hat{X}[\ell \,|\, \ell] \\
\Sigma[\ell+1 \,|\, \ell] &= \bar{F}[\ell]\Sigma[\ell \,|\, \ell]\bar{F}[\ell]^\top + G[\ell]Q[\ell]G^\top[\ell]
\end{aligned}
$$