ECE531 Screencast 7.5: Steady-State Kalman Filter Performance

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Time Invariant Linear Dynamical Model

Suppose you have a time-invariant system with

$$\begin{aligned} F[n] &\equiv F \\ G[n] &\equiv G \\ H[n] &\equiv H \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}\{U[k]\} &= 0 \\ \mathrm{E}\{U[k]U^\top[j]\} &= \begin{cases} Q & k=j \\ 0 & \text{otherwise} \end{cases} \\ \mathrm{E}\{V[k]\} &= 0 \\ \mathrm{E}\{V[k]V^\top[j]\} &= \begin{cases} R & k=j \\ 0 & \text{otherwise} \end{cases} \\ \mathrm{E}\{U[k]V^\top[j]\} &= 0 \qquad \text{for all } j \text{ and } k \end{aligned}$$

Kalman Filter Covariance Recursion

Under our time-invariant assumption, the Kalman filter recursion for the covariance matrices can be written as

$$
\begin{aligned}
K[\ell] &= \Sigma[\ell\,|\,\ell-1]H^\top \left(H\Sigma[\ell\,|\,\ell-1]H^\top + R\right)^{-1} \\
\Sigma[\ell\,|\,\ell] &= \Sigma[\ell\,|\,\ell-1] - K[\ell]H\Sigma[\ell\,|\,\ell-1] \\
\Sigma[\ell+1\,|\,\ell] &= F\Sigma[\ell\,|\,\ell]F^\top + GQG^\top
\end{aligned}
$$

Note that, given $\Sigma[0|-1]$, these matrices can be pre-computed for $\ell = 0, 1, \dots$ (they are not a function of the measurements).

The diagonal elements of these covariance matrices tell us how accurate our estimates and predictions are.

What happens as $\ell \to \infty$?

Kalman Filter Covariance Recursion

If $\{F, H\}$ is completely observable and $\{F, D^\top\}$ is completely controllable, where $Q = D^\top D$, then the covariance matrices and the Kalman gain will converge to a steady state. Let's let

$$K = \lim_{\ell \to \infty} K[\ell]$$
$$S = \lim_{\ell \to \infty} \Sigma[\ell \,|\, \ell]$$
$$P = \lim_{\ell \to \infty} \Sigma[\ell + 1 \,|\, \ell]$$

Then, in the limit, we have

$$\begin{aligned} K &= PH^\top \left(HPH^\top + R\right)^{-1} \\ S &= P - KHP \\ P &= FSF^\top + GQG^\top \end{aligned}$$

Discrete-Time Algebraic Riccati Equation

Since $P = FSF^\top + GQG^\top$, we can substitute in for S and K to write

$$\boxed{P = F[P - PH^\top(HPH^\top + R)^{-1}HP]F^\top + GQG^\top}$$

This is called the discrete-time algebraic Riccati equation. Everything in here is known except P. We want to solve for P.

If all quantities are scalar, it is not difficult to solve for P since we have

$$P = F^2\left[P - \frac{P^2H^2}{H^2P + R}\right] + G^2Q$$

which can be written as a quadratic equation in P and P will have a unique positive solution.

Once you've determined P you can get K and S from

$$\begin{aligned} K &= PH^\top\left(HPH^\top + R\right)^{-1} \\ S &= P - KHP \end{aligned}$$

Discrete-Time Algebraic Riccati Equation

$$\boxed{P = F[P - PH^\top(HPH^\top + R)^{-1}HP]F^\top + GQG^\top}$$

If P is not scalar, the discrete-time algebraic Riccati equation usually requires numerical solution. This can be done with Matlab function `dare`.

For example, suppose we have one-dimensional motion with $T = 0.1$

$$X[n+1] = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} X[n] + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} U[n] \qquad Y[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} X[n] + V[n]$$

and with noise covariances $R = 0.01$ and $Q = 1$. We can call `dare(F',H',G*Q*G',R)` to get

$$P = \begin{bmatrix} 0.0057 & 0.0125 \\ 0.0125 & 0.0553 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0.0036 & 0.0080 \\ 0.0080 & 0.0453 \end{bmatrix}$$

P_{11} (S_{11}) and P_{22} (S_{22}) tell us the steady-state prediction (estimation) variance for the particle's position and velocity, respectively.