ECE531 Screencast 8.3: Detection with a Finite Number of Possible Observations

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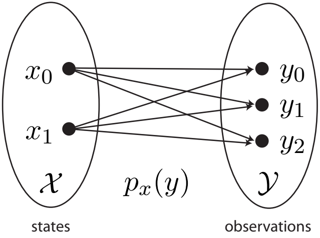
Finite Observation Sets: Conditional Probability Matrix



When $\mathcal{X}$ and $\mathcal{Y}$ are finite with $|\mathcal{X}| = N$ and $|\mathcal{Y}| = L$, we can conveniently represent the conditional/parameterized probabilities $p_x(y)$ in matrix form:

$$P = \begin{bmatrix} p_{x=x_0}(y=y_0) & \cdots & p_{x=x_{N-1}}(y=y_0) \\ \vdots & \ddots & \vdots \\ p_{x=x_0}(y=y_{L-1}) & \cdots & p_{x=x_{N-1}}(y=y_{L-1}) \end{bmatrix} \in \mathbb{R}^{L \times N}$$

Finite Observation Sets: Conditional Decision Probabilities

Recall our decision matrix notation with $D \in \mathbb{R}^{M \times L}$. Let

$$T = DP \in \mathbb{R}^{M \times N}$$

Note that

$$\begin{aligned} T_{ij} &= \sum_{k=0}^{L-1} D_{ik} P_{kj} \\ &= \sum_{k=0}^{L-1} D_{ik} \mathrm{Prob}(y = y_k | x = x_j) \end{aligned}$$

Interpretation: T_{ij} is the probability of deciding $\mathcal{H}_i$ when the state is x_j.

Finite Observation Sets: Decision Costs

- Our goal is to specify a decision rule that is optimum in some sense.
- To do this, we specify a matrix C of **decision costs** where C_{ij} is the cost of deciding $\mathcal{H}_i$ when the state is x_j.

Examples:

1. Uniform cost assignment (UCA)

$$C_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

2. Quadradic cost assignment ($M = N$ and $\mathcal{X}$ is a subset of $\mathbb{R}$)

$$C_{ij} = (x_i - x_j)^2$$

Finite Observation Sets: Conditional Risks

Notation:

- $t_j \in \mathbb{R}^M =$ jth column of $T = DP$. This column contains the probabilities of deciding $\mathcal{H}_0, \dots, \mathcal{H}_{M-1}$ when the state is x_j.
- $c_j \in \mathbb{R}^M =$ jth column of cost matrix C. This column contains the costs of deciding $\mathcal{H}_0, \dots, \mathcal{H}_{M-1}$ when the state is x_j.
- $p_j \in \mathbb{R}^L =$ jth column of conditional probability matrix P. This column contains the probabilities of observing $y_0, \dots, y_{L-1}$ when the state is x_j.

Note that the inner product

$$R_j(D) = c_j^\top t_j = c_j^\top D p_j \qquad j \in \{0, \dots, N-1\}$$

gives the expected cost (also called the **conditional risk**) of using the decision matrix D when the state is x_j.

Example

Suppose we have two states and three possible observations. Let

$$P = \begin{bmatrix} 0.9 & 0 \\ 0.1 & 0.1 \\ 0 & 0.9 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and assume the UCA. Compute the conditional risks.

We can write

$$T = DP = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.9 \end{bmatrix} = \begin{bmatrix} t_0 & t_1 \end{bmatrix}$$

and since

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c_0 & c_1 \end{bmatrix}$$

we have the conditional risks $R_0(D) = c_0^\top t_0 = 0$ and $R_1(D) = c_1^\top t_1 = 0.1$.