ECE531 Screencast 8.4: An Example of Setting up Conditional Risks in the Finite Observation Model

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Coin Flipping Working Example: Part 1

Scenario

We have a scenario with n i.i.d. coin flips where a H occurs with probability q and a T occurs with probability $1-q$. The parameter q takes one of two possible values $0 \le q_0 < q_1 \le 1$.

- The observation is the number of heads.
- We want to decide between $\mathcal{H}_0 : q = q_0$ or $\mathcal{H}_1 : q = q_1$.

- The set of states $\mathcal{X} = \{x_0 : q = q_0, x_1 : q = q_1\}$. $N = |\mathcal{X}| = 2$.
- The observation space $\mathcal{Y} = \{0, \dots, n\}$ with

$$p_j(y = k) \quad = \quad \binom{n}{k} q_j^k (1 - q_j)^{n-k}$$

  $L = |\mathcal{Y}| = n + 1$.
- This is a simple binary hypothesis testing problem since $M = N = 2$.

Coin Flipping Working Example: Part 2

Suppose we have $n = 3$ coin flips. Then

$$P = \begin{bmatrix} (1-q_0)^3 & (1-q_1)^3 \\ 3q_0(1-q_0)^2 & 3q_1(1-q_1)^2 \\ 3q_0^2(1-q_0) & 3q_1^2(1-q_1) \\ q_0^3 & q_1^3 \end{bmatrix}$$

Suppose also that we use the uniform cost assignment. Then

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note that there are a finite number of (deterministic) decision matrices that we can consider:

$$D \in \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

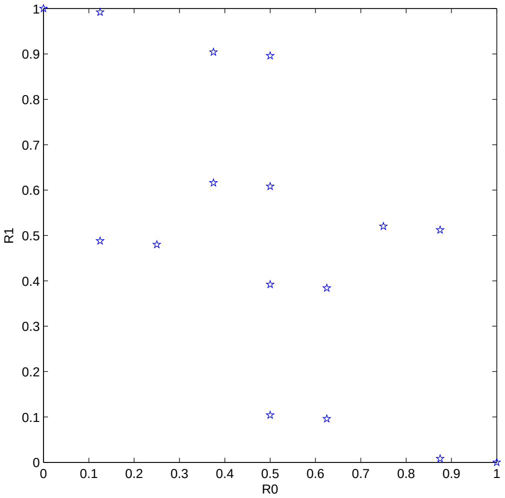
Coin Flipping Working Example: Part 3

We can group the conditional risks $R_j(D)$ into an N-vector

$$R(D) = \begin{bmatrix} R_0(D) \\ R_1(D) \end{bmatrix} = \begin{bmatrix} c_0^\top D p_0 \\ c_1^\top D p_1 \end{bmatrix}$$

- $R(D) \in \mathbb{R}^N$ is called the **conditional risk vector** (CRV).
- Ideally, we would like both $R_0(D)$ and $R_1(D)$ to be small. It is usually not possible, however, to find a D that minimizes both simultaneously.
- To see this, we can plot the coordinates of these conditional risk vectors in $\mathbb{R}^2$ for each of the (deterministic) decision rules...

Conditional Risk Vectors [$q_0 = 0.5$ and $q_1 = 0.8$]



The Problem With Deterministic Decision Rules

- When the observation space is finite, there are only a finite number of deterministic decision matrices and achievable CRVs (M^L).
- In our working example, what if we wanted to balance the risk such that $R_0(D) = R_1(D) = 0.4$?