ECE531 Screencast 8.5: Randomized Decision Rules

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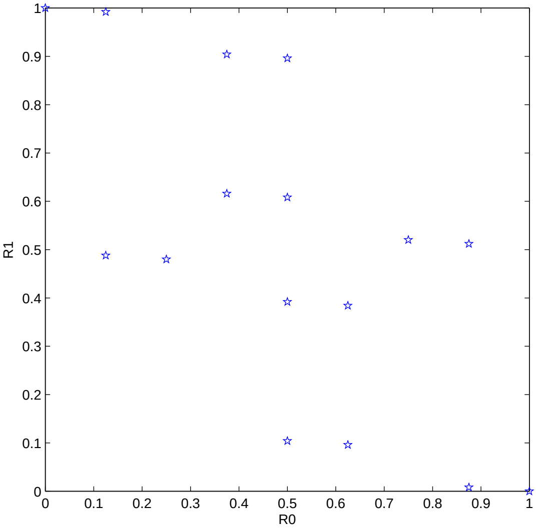
Randomized Decision Rules

- So far, we have focused on deterministic decision rules. Given an observation $y \in \mathcal{Y}$, a deterministic decision rule is a map from $\mathcal{Y}$ directly to $\mathcal{Z}$ (the indices of the hypotheses).
- A generalization of this idea is a **randomized decision rule**. Given an observation $y \in \mathcal{Y}$, a randomized decision rule is a mapping from $\mathcal{Y}$ to a distribution (a pmf) on $\mathcal{Z}$. The set of valid pmfs on $\mathcal{Z}$ is denoted as $\mathcal{P}_M$.
- Examples of random decision matrices:

$$D = \begin{bmatrix} 0.9 & 0.9 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.9 & 0.9 \end{bmatrix} \text{ or } D = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}$$

- Note that the elements of D must be non-negative and the columns must sum to one.
- Also note that the deterministic decision rules are special cases in the family of randomized decision rules $\mathcal{D}$.

Achievable Conditional Risk Vectors

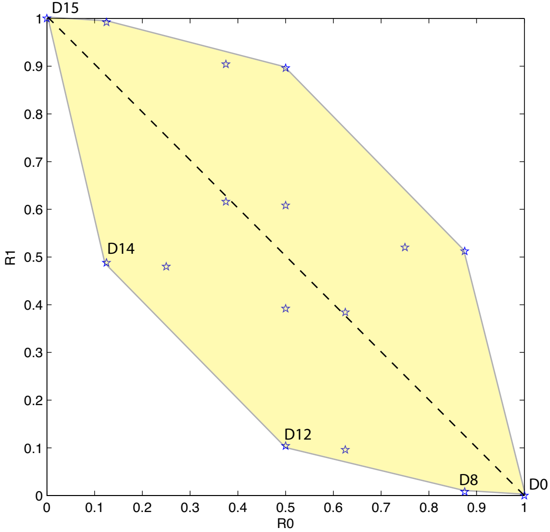


As D ranges over all possible decision rules in $\mathcal{D}$, $R(D)$ traces out a set $\mathcal{Q}$ of **achievable conditional risk vectors**.

$\mathcal{Q}$ is the convex hull of the conditional risk vectors of the deterministic decision rules.

If $|Y|$ is finite, then $\mathcal{Q}$ is a closed and compact polytope in $\mathbb{R}^N$.

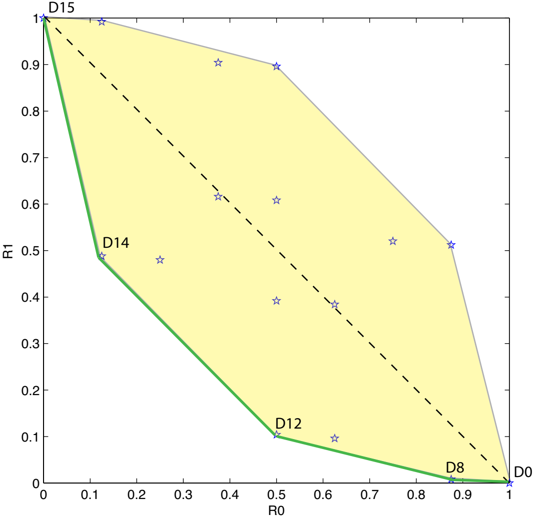
Working Example: Risk Vectors [$q_0 = 0.5$ and $q_1 = 0.8$]



- Can we now balance the risk $R_0 = R_1 = 0.4$?
- What does the line $R_0 + R_1 = 1$ represent? Random guessing.
- Where are the "good" decision rules? Southwest of the random guess line.
- What point on the Southwest boundary of $\mathcal{Q}$ corresponds to the best decision rule?

Optimal Tradeoff Surface of $\mathcal{Q}$

The **optimal tradeoff surface** of $\mathcal{Q}$ is the set of all $R(D)$ for D Pareto optimal. Any "best" decision rule must have a CRV on this optimal tradeoff surface.



Specifying a Unique Decision Rule

Note that the optimal tradeoff surface does not specify a unique best decision rule. An additional criterion is needed.

1. **Neyman Pearson criterion**: Find D that minimizes $R_1(D)$ subject to an upper bound on $R_0(D)$.
2. **Bayes criterion**: Fix some $\lambda \in [0,1]$ and define the weighted Bayes risk $r(D,\lambda) = (1-\lambda)R_0(D) + \lambda(R_1(D))$. Find D that minimizes $r(D,\lambda)$.
3. **Minimax criterion**: Find D that minimizes $\max\{R_0(D), R_1(D)\}$.

Working Example: Risk Vectors [$q_0 = 0.5$ and $q_1 = 0.8$]