ECE531 Screencast 9.1: Detection with an Infinite Number of Possible Observations

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Hypothesis Testing with Infinite Observation Spaces

Lots of real-world problems have observation sets with an infinite number of possibilities. For example:

1. Communications: We transmit a binary symbol $s \in \{s_0, s_1\}$ and the signal is received in additive white Gaussian noise

$$y = s + w$$

   with $w \sim \mathcal{N}(0, \sigma^2)$. The observation $y \in \mathbb{R} = \mathcal{Y}$.

2. Drug testing: A test provides values for the level of testosterone and red blood cell count. The observation $y \in \mathbb{R}^2 = \mathcal{Y}$.

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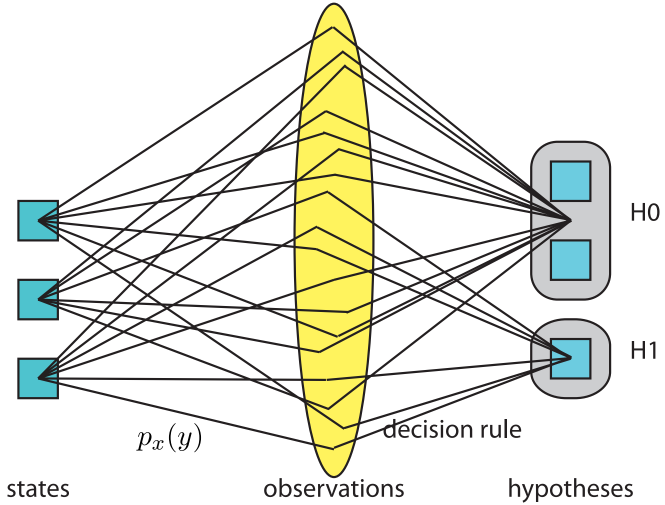
In the case of finite observation spaces, we previously developed the concept of **conditional risk vectors** $R(D) = [R_0(D), \ldots, R_{N-1}(D)]^\top$ with

$$R_j(D) = c_j^\top t_j = c_j^\top D p_j \text{ (finite observation spaces)}$$

We will now extend this concept to **infinite observation spaces**.

Model Summary



Infinite Observation Sets: General Notation

We can generalize our insight from the finite observation space as follows:

1. We denote $\rho_i(y) \in [0,1]$ as the probability of deciding $\mathcal{H}_i$ when the observation is y.
2. Our randomized decision rule is denoted as $\rho(y) = [\rho_0(y), \dots, \rho_{M-1}(y)]^\top : \mathcal{Y} \mapsto \mathcal{P}_M$ where $\mathcal{P}_M \subset \mathbb{R}^M$ is the set of pmfs on $\mathcal{Z}$. We still use $\mathcal{D}$ to denote the set of all possible random/deterministic decision rules $\rho \in \mathcal{D}$.

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3. The cost of deciding $\mathcal{H}_i$ when the state is x_j is still denoted as C_{ij}. Hence, when we start in state x_j and receive the observation y, the cost of using decision rule $\rho(y)$ is

$$C_j(\rho(y)) = \sum_{i=0}^{M-1} \rho_i(y) C_{ij} = c_j^\top \rho(y)$$

Infinite Observation Sets: Conditional Risks

To compute the conditional risk, we must average the cost over the observations (fixing state x_j). The conditional risk for state x_j is then

$$R_j(\rho) = \int_{y \in \mathcal{Y}} C_j(\rho(y)) p_j(y)\, dy = \int_{y \in \mathcal{Y}} \left[\sum_{i=0}^{M-1} \rho_i(y) C_{ij} \right] p_j(y)\, dy$$

where $p_j(y)$ is the known conditional density that probabilistically describes the relationship between state x_j and the observations.

As before, we can group these individual conditional risks into a conditional risk vector $R(\rho) = [R_0(\rho), \dots, R_{N-1}(\rho)]^\top \in \mathbb{R}^N$.

Infinite Observation Sets: Achievable CRVs

If we let the decision rule ρ range over all of $\mathcal{D}$, $R(\rho)$ traces out the set $\mathcal{Q}$ of achievable conditional risk vectors in $\mathbb{R}^N$.

Theorem

$\mathcal{Q}$ is a closed and bounded convex subset of $\mathbb{R}^N$.

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Key point: The concepts of **Pareto optimal decision rules** and the **optimal tradeoff surface** of $\mathcal{Q}$ also apply to the case of infinite $\mathcal{Y}$.

Note that $\mathcal{Q}$ is probably not a polytope anymore.

Summary of Main Results

Conditional risks as a way of quantifying the performance/consequences of a decision rule when the state is x_j:

$$R_j(D) = c_j^\top D p_j \text{ (finite observation spaces)}$$

$$R_j(\rho) = \int_{y \in \mathcal{Y}} \left[\sum_{i=0}^{M-1} \rho_i(y) C_{ij} \right] p_j(y)\, dy \text{ (infinite observation spaces)}$$

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The conditional risks for finite and infinite $\mathcal{Y}$ are conceptually similar:

- Both are an inner product of the cost-weighted decision rule and the conditional observation probabilities
- Both yield a set of achievable CRVs that is closed, bounded, and convex
- Convexity implies that minimizing all conditional risks simultaneously is impossible. The conditional risks must be traded off against each other on the optimal tradeoff surface.