ECE531 Screencast 9.2: N-P Detection with an Infinite Number of Possible Observations

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Neyman Pearson Hypothesis Testing: Finite/Infinite Obs.

Finite number of possible observations:

- Compute the likelihood ratios

$$L_\ell = \frac{P_{\ell,1}}{P_{\ell,0}}$$

- Pick the threshold v to be the smallest value such that

$$P_{\mathsf{fp}} = \sum_{\ell : L_\ell > v} P_{\ell,0} \leq \alpha$$

- If $P_{\mathsf{fp}}(\delta^v) < \alpha$, compute the randomization γ so that $P_{\mathsf{fp}} = \alpha$.
- The N-P decision rule is then

$$\rho^{\mathsf{NP}}(y_\ell) = \begin{cases} 1 & L_\ell > v \\ \gamma & L_\ell = v \\ 0 & L_\ell < v \end{cases}$$

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Infinite number of possible observations:

- Compute the likelihood ratios

$$L(y) = \frac{p_1(y)}{p_0(y)}$$

- Pick the threshold v to be the smallest value such that

$$P_{\mathsf{fp}} = \int_{y : L(y) > v} p_0(y)\, dy \le \alpha$$

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Example: Coherent Detection of BPSK

Suppose a transmitter sends one of two scalar signals a_0 or a_1 and the signal arrives at a receiver corrupted by zero-mean additive white Gaussian noise (AWGN) with variance σ^2.

Suppose we have a scalar observation. Signal model when $x_j = a_j$:

$$Y = a_j + \eta$$

where a_j is the scalar signal and $\eta \sim \mathcal{N}(0, \sigma^2)$. Hence

$$p_j(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y - a_j)^2}{2\sigma^2}\right)$$

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We want to use N-P hypothesis testing to maximize

$$P_D = \mathrm{Prob}(\text{decide } \mathcal{H}_1 \,|\, a_1 \text{ was sent})$$

subject to the constraint

$$P_{\mathsf{fp}} = \mathrm{Prob}(\text{decide } \mathcal{H}_1 \,|\, a_0 \text{ was sent}) \leq \alpha.$$

Example: Coherent Detection of BPSK

The N-P decision rule will be of the form

$$\rho^{\mathsf{NP}}(y) = \begin{cases} 1 & \text{if } L(y) > v \\ \gamma & \text{if } L(y) = v \\ 0 & \text{if } L(y) < v \end{cases}$$

where $v \geq 0$ and $0 \leq \gamma(y) \leq 1$ are such that $P_{\mathsf{fp}}(\rho^{\mathsf{NP}}) = \alpha$.

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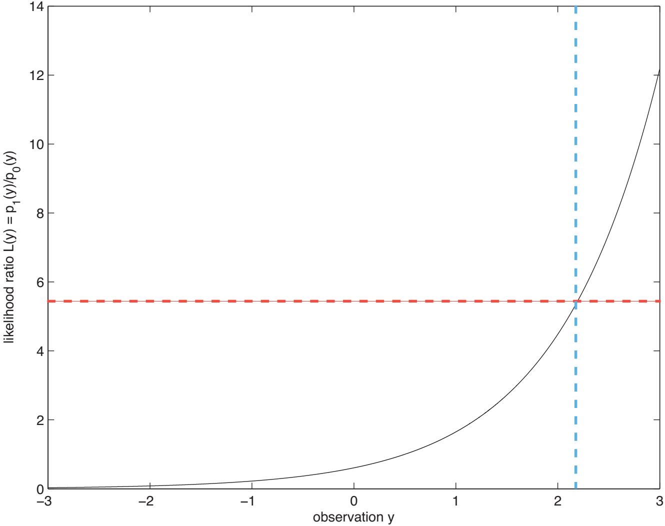
where $v \geq 0$ and $0 \leq \gamma(y) \leq 1$ are such that $P_{\mathsf{fp}}(\rho^{\mathsf{NP}}) = \alpha$.

We need to find the smallest v such that

$$\int_{\mathcal{Y}_v} p_0(y)\, dy \leq \alpha$$

where $\mathcal{Y}_v = \{y \in \mathcal{Y} : L(y) > v\}$.

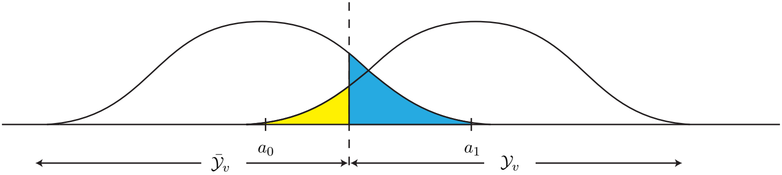
Example: Likelihood Ratio for $a_0 = 0$, $a_1 = 1$, $\sigma^2 = 1$



Example: Coherent Detection of BPSK

Note that, since $a_1 > a_0$, the likelihood ratio $L(y) = \frac{p_1(y)}{p_0(y)}$ is monotonically increasing. This means that finding v is equivalent to finding a threshold τ so that

$$\int_\tau^\infty p_0(y)\,dy \le \alpha \quad \Leftrightarrow \quad Q\left(\frac{\tau - a_0}{\sigma}\right) \le \alpha \quad \Leftrightarrow \quad \tau \ge \sigma Q^{-1}(\alpha) + a_0$$



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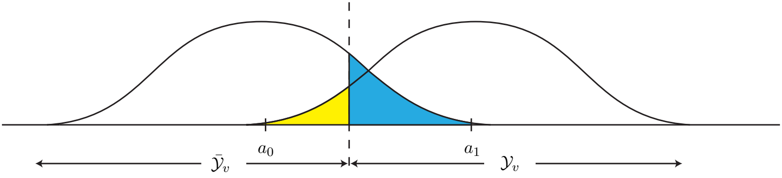

The N-P decision rule is then

$$\rho^{\text{NP}}(y) = \begin{cases} 1 & \text{if } y \ge \sigma Q^{-1}(\alpha) + a_0 \\ 0 & \text{if } y < \sigma Q^{-1}(\alpha) + a_0 \end{cases}$$

and $P_D = \int_{\sigma Q^{-1}(\alpha)+a_0}^{\infty} p_1(y)\,dy$.

Final Comments on Neyman-Pearson Hypothesis Testing

1. N-P decision rules are useful in asymmetric risk scenarios or in scenarios where one has to guarantee a certain probability of false detection.
2. N-P decision rules are always based on simple likelihood ratio comparisons. The comparison threshold is chosen to satisfy the significance level constraint.
3. Randomization may be necessary for N-P decision rules. Without randomization, the power of the test may not be maximized for the significance level constraint.
4. The original N-P paper: "On the Problem of the Most Efficient Tests of Statistical Hypotheses," J. Neyman and E.S. Pearson, *Philosophical Transactions of the Royal Society of London, Series A, Containing Papers of a Mathematical or Physical Character*, Vol. 231 (1933), pp. 289-337. Available on jstor.org.