ECE531 Screencast 9.6: Bayesian Detection Example

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Example: Coherent Detection of BPSK

Suppose a transmitter sends one of two scalar signals and the signals arrive at a receiver corrupted by zero-mean additive white Gaussian noise (AWGN) with variance σ^2. We want to use Bayesian hypothesis testing to determine which signal was sent.

Signal model conditioned on state x_j:

$$Y = a_j + \eta$$

where a_j is the scalar signal and $\eta \sim \mathcal{N}(0, \sigma^2)$. Hence

$$p_j(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y-a_j)^2}{2\sigma^2}\right)$$

Hypotheses:

$$\begin{aligned} \mathcal{H}_0 &: a_0 \text{ was sent, or } Y \sim \mathcal{N}(a_0, \sigma^2) \\ \mathcal{H}_1 &: a_1 \text{ was sent, or } Y \sim \mathcal{N}(a_1, \sigma^2) \end{aligned}$$

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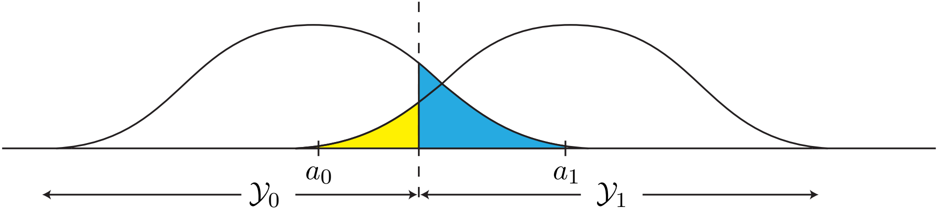
This is just a simple binary hypothesis testing problem. Under the UCA, we can write

$$
\begin{aligned}
R_0(\rho) &= \int_{y \in \mathcal{Y}_1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y-a_0)^2}{2\sigma^2} \right) dy \\
R_1(\rho) &= \int_{y \in \mathcal{Y}_0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y-a_1)^2}{2\sigma^2} \right) dy
\end{aligned}
$$

where $\mathcal{Y}_j = \{y \in \mathcal{Y} : \rho_j(y) = 1\}$. Intuitively,



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Given a prior π_0, $\pi_1 = 1 - \pi_0$, we can write the Bayes risk as

$$r(\rho, \pi) \quad = \quad \pi_0 R_0(\rho) + (1 - \pi_0) R_1(\rho)$$

The Bayes decision rule can be found by computing the likelihood ratio and comparing it to the threshold τ:

$$\begin{aligned} L(y) = \frac{p_1(y)}{p_0(y)} > \tau \quad & \Leftrightarrow \quad \frac{\exp\left(\frac{-(y-a_1)^2}{2\sigma^2}\right)}{\exp\left(\frac{-(y-a_0)^2}{2\sigma^2}\right)} > \frac{\pi_0}{\pi_1} \\ & \Leftrightarrow \quad \frac{(y-a_0)^2 - (y-a_1)^2}{2\sigma^2} > \ln\frac{\pi_0}{\pi_1} \\ & \Leftrightarrow \quad y > \frac{a_0 + a_1}{2} + \frac{\sigma^2}{a_1 - a_0} \ln\frac{\pi_0}{\pi_1} \end{aligned}$$

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Let $\psi := \frac{a_0+a_1}{2} + \frac{\sigma^2}{a_1-a_0} \ln \frac{\pi_0}{\pi_1}$. Our Bayes decision rule is then:

$$\delta^{B\pi}(y) = \begin{cases} 1 & \text{if } y > \psi \\ 0/1 & \text{if } y = \psi \\ 0 & \text{if } y < \psi. \end{cases}$$

Using this decision rule, the Bayes risk is then

$$r(\delta^{B\pi}, \pi) \quad = \quad \pi_0 Q\left(\frac{\psi - a_0}{\sigma}\right) + (1-\pi_0) Q\left(\frac{a_1 - \psi}{\sigma}\right)$$

where $Q(x) := \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$. Remarks:

- When $\pi_0 = \pi_1 = \frac{1}{2}$, the decision boundary is simply $\frac{a_0+a_1}{2}$, i.e. the midpoint between a_0 and a_1 and $r(\delta^{B\pi}, \pi) = Q\left(\frac{a_1-a_0}{2\sigma}\right)$.
- When σ is very small, the prior has little effect on the decision boundary. When σ is very large, the prior becomes more important.